

Regularization just by quantization — a new approach to the old problem of infinities in quantum field theory

(Draft of lecture notes - Part 1)

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Abstract

This is the first part of lecture notes on my approach to field quantization. I explain on a simple scalar-field model the physical motivation and show some preliminary applications (field produced by a pointlike charge and the vacuum-to-vacuum loop diagram). The approach is based on an appropriate construction of a representation of harmonic-oscillator Lie algebra.

Sensible mathematics involves neglecting a quantity when it is small — not neglecting it just because it is infinitely great and you do not want it!

P. A. M. Dirac (1975)[1]

For many the infinities that still haunt physics cry for further and deeper quantization, but there has been little agreement on exactly what and how far to quantize.

D. R. Finkelstein (2006) [2]

I. INTRODUCTION

When solving problems in quantum mechanics one often notices that discrete energy levels are determined by practically one mathematical condition — finite norm of eigenvectors. The reason for the finiteness is simple — squared norm represents sums of probabilities associated with maximal sets of jointly measurable physical quantities. It should be normalizable to 1 and thus cannot be infinite. Finiteness of another important quantity, $\langle\psi|H^2|\psi\rangle$, guarantees that fluctuations $\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2}$ are finite. From a mathematical point of view $\langle\psi|H^2|\psi\rangle < \infty$ means that $|\psi\rangle$ and $H|\psi\rangle$ belong to the same Hilbert space, a condition that defines the domain of H . As we can see, the apparently trivial “law of finiteness of quantities that should be finite” is raised in quantum mechanics to the level of important technical principle. It determines values of physical quantities (spectra of operators) and sets of acceptable states of physical systems (domains of operators). If one allowed for discrete spectra and infinite norms then harmonic oscillators would possess infinitely many arbitrarily large negative energy levels.

In quantum field theory we become much more tolerant. Infrared and ultraviolet “catastrophes” (infinities occurring at various stages of calculations) are regularized and renormalized. Algorithms for extracting physically reasonable numbers from infinite theoretical predictions were invented and resulted in several Nobel Prizes in physics. The procedures generally seem to work, but how to explain to a beginner why we cannot proceed in a similar way in standard quantum mechanics with “divergent probabilities associated with non-normalizable eigenvectors”? We could take the inverted Gaussian associated with $E = -\hbar\omega/2$, regularize the divergent norm by a cutoff, divide the eigenvector by such a regularized norm, compute average energy, and finally remove the cutoff. The procedure

would certainly imply some kind of “catastrophe”, but basically what would be wrong with it? A similar level of logical rigor seems acceptable in quantum field theory.

Yet, in spite of all the successes of quantum field theory I’m convinced that it is the paradigm of quantum mechanics that is physically correct. We have to determine mathematical structures by the well-definiteness of physical quantities. Not all mathematically well defined models are physical, but models that are mathematically inconsistent are unphysical for sure. The fact that it is practically impossible to perform a nontrivial quantum field theoretic calculation without encountering an ill defined mathematical expression shows, in my opinion, that the most fundamental theory of physics is formulated in a wrong way.

What I write is, of course, not original. Dirac repeatedly stressed that a deep change is needed in conceptual foundations of field quantization. In his opinion quantum field theory is still at the stage analogous to atomic physics before the advent of Schrödinger’s equation — we had agreement between theory and experiment (Bohr’s model of atom predicted correct spectra), but theoretical ideas were completely wrong. In his last two papers [3, 4], published in the year of his death, he formulated his desiderata for fundamental physical theories. He believed that quantum dynamics should always be described by Heisenberg equation of motion with appropriate Hamiltonian. Modifications are expected in algebraic properties of dynamical variables since here freedom is immense. He stressed the unexplored potential of reducible representations of fundamental symmetries.

In this context it should be reminded that the Heisenberg equation is completely unrelated to classical Euler-Lagrange equations. Links with the latter can only be established after having decided what are the commutation relations between the dynamical variables, and which representation to choose.

So, which algebra should be satisfied by the dynamical variables?

Finkelstein suggests [2, 5] that one should quantize in terms of irreducible representations of those Lie algebras whose physically meaningful representations are finite dimensional. In these notes I want to concentrate on an approach which is, in a sense, complementary to the philosophy of Finkelstein (I prefer reducible but infinitely-dimensional representations), and whose main ideas can be found already in the two preliminary papers of mine [6, 7]. However, copyright for the term “regularization by quantization” is due to David Finkelstein.

Restricting the discussion to spinless fields I propose to consider the following two options:

Canonical commutation relations (CCR),

$$[a(\mathbf{p}), a(\mathbf{p}')^\dagger] = \delta(\mathbf{p}, \mathbf{p}'), \quad (1)$$

and harmonic oscillator Lie algebra (HOLA), whose nontrivial commutators read

$$[a(\mathbf{p}), a(\mathbf{p}')^\dagger] = \delta(\mathbf{p}, \mathbf{p}') I(\mathbf{p}), \quad (2)$$

$$[a(\mathbf{p}), n(\mathbf{p}')] = \delta(\mathbf{p}, \mathbf{p}') a(\mathbf{p}), \quad (3)$$

$$[a(\mathbf{p})^\dagger, n(\mathbf{p}')] = -\delta(\mathbf{p}, \mathbf{p}') a(\mathbf{p})^\dagger, \quad (4)$$

with $\delta(\mathbf{p}, \mathbf{p}')$ playing the role of structure constants.

The idea of replacing Poisson brackets by commutators leads to (1) as a natural candidate. An interpretation of fields as ensembles of harmonic oscillators suggests (2)–(4). The usual identifications $n(\mathbf{p}) = a(\mathbf{p})^\dagger a(\mathbf{p})$ and $I(\mathbf{p}) = 1$ may seem to imply that CCR determines the form of HOLA, but this is not the case, as we shall see in a moment. Moreover, whatever algebra one selects, one yet has to tell something about $\delta(\mathbf{p}, \mathbf{p}')$, which is less obvious than one might expect.

II. CCR DOES NOT DETERMINE HOLA

Let us now show that a concrete representation of CCR does not yet fix the form of an associated HOLA. Let $[a_1, a_1^\dagger] = I_1$ be the representation of CCR in a Hilbert space \mathcal{H}_1 (I_1 is the identity map in \mathcal{H}_1). Defining $n_1 = a_1^\dagger a_1$ we obtain a representation of HOLA:

$$[a_1, a_1^\dagger] = I_1, \quad (5)$$

$$[a_1, n_1] = a_1, \quad (6)$$

$$[a_1^\dagger, n_1] = -a_1^\dagger. \quad (7)$$

Now take

$$a_2 = \frac{1}{\sqrt{2}} (a_1 \otimes I_1 + I_1 \otimes a_1), \quad (8)$$

$$a_2^\dagger = \frac{1}{\sqrt{2}} (a_1^\dagger \otimes I_1 + I_1 \otimes a_1^\dagger). \quad (9)$$

One checks that

$$[a_2, a_2^\dagger] = I_1 \otimes I_1 = I_2, \quad (10)$$

i.e. this is a representation of CCR in $\mathcal{H}_2 = \mathcal{H}_1 \otimes \mathcal{H}_1$. In order to extend it to HOLA we have to find an appropriate n_2 . Obviously, one choice is

$$n_2 = a_2^\dagger a_2 \quad (11)$$

$$= \frac{1}{2} \left(a_1^\dagger a_1 \otimes I_1 + I_1 \otimes a_1^\dagger a_1 + a_1^\dagger \otimes a_1 + a_1 \otimes a_1^\dagger \right). \quad (12)$$

The Hamiltonian $H_2 = \hbar\omega n_2$ represents a system of two *interacting* oscillators of frequency $\omega/2$, where

$$H_2^{(0)} = \frac{\hbar\omega}{2} a_1^\dagger a_1 \otimes I_1 + I_1 \otimes \frac{\hbar\omega}{2} a_1^\dagger a_1 \quad (13)$$

is the free part. The interaction term

$$H_2^{(1)} = \frac{\hbar\omega}{2} \left(a_1^\dagger \otimes a_1 + a_1 \otimes a_1^\dagger \right) \quad (14)$$

is responsible for energy exchange. The algebra is as it should be

$$[a_2, a_2^\dagger] = I_2, \quad (15)$$

$$[a_2, n_2] = a_2, \quad (16)$$

$$[a_2^\dagger, n_2] = -a_2^\dagger. \quad (17)$$

The problem is that n_2 could be replaced by

$$\tilde{n}_2 = n_1 \otimes I_1 + I_1 \otimes n_1, \quad (18)$$

still satisfying the same algebra

$$[a_2, a_2^\dagger] = I_2, \quad (19)$$

$$[a_2, \tilde{n}_2] = a_2, \quad (20)$$

$$[a_2^\dagger, \tilde{n}_2] = -a_2^\dagger, \quad (21)$$

with the same a_2 , a_2^\dagger and I_2 . Physically the new representation corresponds to two *non-interacting* oscillators of frequency ω if we employ the same correspondence between Hamiltonian and the number-of-excitations operator:

$$\tilde{H}_2 = \hbar\omega \tilde{n}_2 \quad (22)$$

$$= \hbar\omega a_1^\dagger a_1 \otimes I_1 + I_1 \otimes \hbar\omega a_1^\dagger a_1. \quad (23)$$

It is not surprising that still further possibilities exist,

$$\tilde{\tilde{n}}_2 = a_1^\dagger \otimes a_1 + a_1 \otimes a_1^\dagger, \quad (24)$$

or an arbitrary convex combination $p_1 \tilde{\tilde{n}}_2 + p_2 \tilde{\tilde{n}}_2$, $p_1 + p_2 = 1$. An example of the latter has already been encountered,

$$n_2 = \frac{1}{2} \tilde{\tilde{n}}_2 + \frac{1}{2} \tilde{\tilde{n}}_2. \quad (25)$$

We will later see that the above ambiguities imply certain level of freedom in definitions of quantum-field 4-momenta and number operators.

III. NUMBER-OF-PARTICLES REPRESENTATION (BOSONIC FOCK SPACE)

Before we discuss field quantization in more detail let me introduce here the familiar number-of-particles representation of multi particle systems. The construction is interesting in itself and is often regarded as *the* representation appropriate for quantum fields. We will later see that the latter statement is not necessarily true, or at least is less obvious than what one is generally led to believe.

One often reads that “there is no quantum field theory, there is only a theory of multi-particle systems”. In the next chapter I will give arguments why formal similarities between multi-particle systems and quantum fields may be misleading, a subtlety closely related to the distinction between particles and quasi-particles. But first we have to understand the standard construction, although in the end it will *not* be employed in my own approach to field quantization.

We begin with a Hilbert space \mathcal{H} whose countable basis will be denoted by $|j\rangle$, $j = 1, 2, \dots$. Hilbert spaces that possess a countable basis are called separable (the standard Hilbert space from the first semester of quantum mechanics is separable, an example of the countable basis being provided by Hermite polynomials). Any operator A has matrix elements $\langle k|A|l\rangle = A_{kl}$ and can be written as

$$A = \sum_k |k\rangle \langle k| A \sum_l |l\rangle \langle l| = \sum_{kl} |k\rangle \langle k| A |l\rangle \langle l| = \sum_{kl} A_{kl} |k\rangle \langle l|. \quad (26)$$

If $|\tilde{j}\rangle$, $j = 1, 2, \dots$ is another basis in \mathcal{H} , then

$$V = \sum_j |\tilde{j}\rangle \langle j| \quad (27)$$

is unitary,

$$VV^\dagger = \sum_j |\tilde{j}\rangle\langle j| \sum_k |k\rangle\langle \tilde{k}| = \sum_{jk} |\tilde{j}\rangle\langle j|k\rangle\langle \tilde{k}| = \sum_{jk} |\tilde{j}\rangle\delta_{jk}\langle \tilde{k}| = \sum_j |\tilde{j}\rangle\langle \tilde{j}| = \mathbb{I}, \quad (28)$$

$$V^\dagger V = \sum_k |k\rangle\langle \tilde{k}| \sum_j |\tilde{j}\rangle\langle j| = \sum_{jk} |k\rangle\langle \tilde{k}|\tilde{j}\rangle\langle j| = \sum_{jk} |k\rangle\delta_{jk}\langle j| = \sum_j |j\rangle\langle j| = \mathbb{I}, \quad (29)$$

relates the two bases,

$$V|k\rangle = \sum_j |\tilde{j}\rangle\langle j|k\rangle = \sum_j |\tilde{j}\rangle\delta_{jk} = |\tilde{k}\rangle, \quad (30)$$

and has matrix elements

$$\langle k|V|l\rangle = \langle k|\sum_j |\tilde{j}\rangle\langle j|l\rangle = \sum_j \langle k|\tilde{j}\rangle\delta_{jl} = \langle k|\tilde{l}\rangle. \quad (31)$$

Moreover,

$$|\tilde{k}\rangle = V|k\rangle = \sum_j |j\rangle\langle j|V|k\rangle = \sum_j V_{jk}|j\rangle \quad (32)$$

shows that numbers V_{jk} allow us to write the new basis as a linear combination of the old one.

Now let us take a countable collection of operators a_j satisfying CCR $[a_k, a_l^\dagger] = \delta_{kl}$, and let $|0\rangle$ be their common “vacuum state”,

$$a_j|0\rangle = 0, \quad j = 1, 2, \dots \quad (33)$$

$$(0|0\rangle = 1. \quad (34)$$

We do not assume that $|0\rangle$ belongs to \mathcal{H} . It is a completely independent object, belonging to some linear space \mathcal{F} , say. Let $|j\rangle = a_j^\dagger|0\rangle$, $(j| = (0|a_j$. The new vectors are orthonormal

$$(k|l) = (0|a_k a_l^\dagger|0\rangle = (0|a_k a_l^\dagger - a_l^\dagger a_k|0\rangle = \delta_{kl}(0|0\rangle) = \delta_{kl}, \quad (35)$$

and (by definition) also belong to \mathcal{F} . $|j\rangle$ are orthogonal to $|0\rangle$,

$$(j|0\rangle = (0|a_j|0\rangle = 0. \quad (36)$$

Taking numbers $V_{kl} = \langle k|V|l\rangle$, defined in (31), we can define a new set of vectors

$$\sum_j |j\rangle\langle j|V|k\rangle = \sum_j V_{jk}|j\rangle = \underbrace{\sum_j V_{jk}a_j^\dagger|0\rangle}_{\tilde{a}_k^\dagger|0\rangle} = |\tilde{k}\rangle, \quad (37)$$

$$(j|\tilde{k}\rangle = \langle j|V|k\rangle = \langle j|\tilde{k}\rangle. \quad (38)$$

Note that

$$\tilde{a}_k = \left(\sum_j V_{jk} a_j^\dagger \right)^\dagger = \sum_j \overline{V_{jk}} a_j = \sum_j V_{kj}^\dagger a_j, \quad (39)$$

so that

$$\begin{aligned} [\tilde{a}_k, \tilde{a}_m^\dagger] &= \left[\sum_j V_{kj}^\dagger a_j, \sum_l V_{lm} a_l^\dagger \right] = \sum_j V_{kj}^\dagger \sum_l V_{lm} [a_j, a_l^\dagger] = \sum_{jl} V_{kj}^\dagger V_{lm} \delta_{jl} \\ &= \sum_j V_{kj}^\dagger V_{jm} = (V^\dagger V)_{km} = \langle k | V^\dagger V | m \rangle = \langle \tilde{k} | \tilde{m} \rangle = \delta_{km}. \end{aligned} \quad (40)$$

The fact that

$$[\tilde{a}_k, \tilde{a}_m^\dagger] = [a_k, a_m^\dagger] = \delta_{km} \quad (41)$$

suggests that there exists a unitary transformation U satisfying

$$U a_k^\dagger U^\dagger = \tilde{a}_k^\dagger = \sum_j V_{jk} a_j^\dagger. \quad (42)$$

I will construct U explicitly and show that

$$U|0\rangle = U^\dagger|0\rangle = |0\rangle, \quad (43)$$

$$U|k\rangle = U a_k^\dagger |0\rangle = U a_k^\dagger U^\dagger |0\rangle = \tilde{a}_k^\dagger |0\rangle = |\tilde{k}\rangle = \sum_j V_{jk} a_j^\dagger |0\rangle = \sum_j V_{jk} |j\rangle. \quad (44)$$

As we can see, U will play in \mathcal{F} a similar role to that of V in \mathcal{H} ,

$$|\tilde{k}\rangle = U|k\rangle = \sum_j V_{jk} |j\rangle, \quad (45)$$

$$|\tilde{k}\rangle = V|k\rangle = \sum_j V_{jk} |j\rangle. \quad (46)$$

Let us now perform the construction of U . To do so we first note that any unitary transformation can be written in an exponential way. Indeed, eigenvalues λ of a unitary operator V , say, satisfy

$$V|\lambda\rangle = \lambda|\lambda\rangle, \quad (47)$$

$$V^{-1}|\lambda\rangle = \lambda^{-1}|\lambda\rangle, \quad (48)$$

$$\langle\lambda|V|\lambda\rangle = \lambda\langle\lambda|\lambda\rangle, \quad (49)$$

$$\overline{\langle\lambda|V|\lambda\rangle} = \langle\lambda|V^\dagger|\lambda\rangle = \bar{\lambda}\langle\lambda|\lambda\rangle = \langle\lambda|V^{-1}|\lambda\rangle = \lambda^{-1}\langle\lambda|\lambda\rangle, \quad (50)$$

$$\bar{\lambda} = \lambda^{-1}. \quad (51)$$

Therefore, $\lambda = e^{i\varphi(\lambda)}$ for some real and unique $\varphi(\lambda) \in [0, 2\pi)$. Let $\Phi = \sum_{\lambda} \varphi(\lambda) |\lambda\rangle\langle\lambda|$. Then,

$$e^{i\Phi} = \sum_{\lambda} e^{i\varphi(\lambda)} |\lambda\rangle\langle\lambda| = \sum_{\lambda} \lambda |\lambda\rangle\langle\lambda| \quad (52)$$

commutes with V and has the same eigenvectors and eigenvalues as V . Accordingly, $V = e^{i\Phi}$ with $\Phi^\dagger = \Phi$.

In particular, there exists X such that $U = e^X$, $U^\dagger = e^{-X}$. It is known that for any X and Y

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \frac{1}{3!} [X, [X, [X, Y]]] + \dots \quad (53)$$

(Of course, provided products and commutators of operators are meaningful, which is not obvious if the operators are unbounded.) In order to prove this, let us consider the family of operators

$$Y(t) = e^{tX} Y e^{-tX}, \quad (54)$$

$$Y(0) = Y. \quad (55)$$

Differentiating (54) we find

$$\frac{d}{dt} Y(t) = X e^{tX} Y e^{-tX} - e^{tX} Y e^{-tX} X = [X, Y(t)]. \quad (56)$$

On the other hand

$$\begin{aligned} \frac{d}{dt} \left(Y + t[X, Y] + \frac{t^2}{2!} [X, [X, Y]] + \frac{t^3}{3!} [X, [X, [X, Y]]] + \dots \right) \\ = [X, Y] + t[X, [X, Y]] + \frac{t^2}{2!} [X, [X, [X, Y]]] + \dots = [X, Y(t)]. \end{aligned} \quad (57)$$

We conclude that

$$\frac{d}{dt} Y(t) = \frac{d}{dt} \left(Y + t[X, Y] + \frac{t^2}{2!} [X, [X, Y]] + \frac{t^3}{3!} [X, [X, [X, Y]]] + \dots \right), \quad (58)$$

where the differentiated objects satisfy the same initial condition at $t = 0$. By uniqueness of solution of a first order linear differential equation one arrives at

$$Y(t) = e^{tX} Y e^{-tX} = Y + t[X, Y] + \frac{t^2}{2!} [X, [X, Y]] + \frac{t^3}{3!} [X, [X, [X, Y]]] + \dots \quad (59)$$

for any t . Putting $t = 1$ we end the proof.

Now take X of the form

$$X = \sum_{kl} a_k^\dagger x_{kl} a_l. \quad (60)$$

Then

$$[X, a_j^\dagger] = \sum_k a_k^\dagger x_{kj}, \quad (61)$$

$$\begin{aligned} [X, [X, a_j^\dagger]] &= \sum_k \sum_{nm} [a_n^\dagger x_{nm} a_m, a_k^\dagger x_{kj}] = \sum_k \sum_{nm} a_n^\dagger x_{nm} \delta_{mk} x_{kj} = \sum_{nm} a_n^\dagger x_{nm} x_{mj} \\ &= \sum_k a_k^\dagger (x^2)_{kj} \end{aligned} \quad (62)$$

Repeating the procedure, we find

$$\begin{aligned} U a_j^\dagger U^\dagger &= a_j^\dagger + [X, a_j^\dagger] + \frac{1}{2!} [X, [X, a_j^\dagger]] + \frac{1}{3!} [X, [X, [X, a_j^\dagger]]] + \dots \\ &= \sum_k a_k^\dagger \left(\delta_{kj} + x_{kj} + \frac{1}{2!} (x^2)_{kj} + \frac{1}{3!} (x^3)_{kj} + \dots \right) \\ &= \sum_k a_k^\dagger (e^x)_{kj}. \end{aligned} \quad (63)$$

It follows that in order to construct U we first have to express V as $V = e^x$ for some antiunitary $x = -x^\dagger$. The matrix elements $x_{kj} = \langle k|x|j \rangle$ are then employed in construction of X satisfying $U = e^X$. Since $X|0\rangle = 0$ we get

$$U|0\rangle = \left(\mathbb{I} + X + \frac{1}{2!} X^2 + \frac{1}{3!} X^3 + \dots \right) |0\rangle = \mathbb{I}|0\rangle = |0\rangle. \quad (64)$$

In consequence, once we have a_j corresponding to some basis $|j\rangle$, we know how to construct \tilde{a}_j such that $|\tilde{j}\rangle$ is related to $|j\rangle$ in a given way, determined by the link between $|\tilde{j}\rangle$ and $|j\rangle$.

The vectors $|j\rangle$ in \mathcal{F} so constructed are completely analogous to $|j\rangle$ in \mathcal{H} . Having any vector

$$|\psi\rangle = \sum_j \psi_j |j\rangle \in \mathcal{H} \quad (65)$$

we can construct

$$|\psi\rangle = \sum_j \psi_j |j\rangle = \underbrace{\sum_j \psi_j a_j^\dagger |0\rangle}_{\hat{\psi}} = \hat{\psi} |0\rangle \in \mathcal{F} \quad (66)$$

whose properties are “identical” to those of $|\psi\rangle$. $\hat{\psi}$ is an example of *field operator*.

It is interesting to check the action of X given by (60) on $\hat{\psi}|0\rangle$,

$$\begin{aligned} X\hat{\psi}|0\rangle &= \sum_{kl} a_k^\dagger x_{kl} a_l \sum_j \psi_j a_j^\dagger |0\rangle = \sum_{klj} a_k^\dagger x_{kl} \psi_j [a_l, a_j^\dagger] |0\rangle = \sum_{klj} a_k^\dagger x_{kl} \psi_j \delta_{lj} |0\rangle \\ &= \sum_{kl} a_k^\dagger x_{kl} \psi_l |0\rangle = \sum_k (x\psi)_k a_k^\dagger |0\rangle = \widehat{x\psi}|0\rangle. \end{aligned} \quad (67)$$

We find $X\hat{\psi}|0\rangle = \widehat{x\psi}|0\rangle$ but this is true only in action on $|0\rangle$. The general rule for X and $\hat{\psi}$ of the above form is

$$[X, \hat{\psi}] = \widehat{x\psi}. \quad (68)$$

If ψ_λ is an eigenvector of x , i.e. $x\psi_\lambda = \lambda\psi_\lambda$, then the \mathcal{F} -space counterpart of the eigenvalue problem is

$$[X, \hat{\psi}_\lambda] = \lambda\hat{\psi}_\lambda, \quad (69)$$

implying

$$[X, \hat{\psi}_{\lambda_1} \dots \hat{\psi}_{\lambda_K}] = (\lambda_1 + \dots + \lambda_K) \hat{\psi}_{\lambda_1} \dots \hat{\psi}_{\lambda_K}, \quad (70)$$

$$X\hat{\psi}_{\lambda_1} \dots \hat{\psi}_{\lambda_K} |0\rangle = (\lambda_1 + \dots + \lambda_K) \hat{\psi}_{\lambda_1} \dots \hat{\psi}_{\lambda_K} |0\rangle. \quad (71)$$

The latter formula illustrates the essential difference between \mathcal{H} and \mathcal{F} , namely the fact that in \mathcal{F} it makes sense to consider vectors of the form

$$|jk\rangle = a_j^\dagger a_k^\dagger |0\rangle, \quad (72)$$

$$|jkl\rangle = a_j^\dagger a_k^\dagger a_l^\dagger |0\rangle, \quad (73)$$

\vdots

all of them orthogonal to $|l\rangle$, $l = 0, 1, 2, \dots$. The operator

$$n = \sum_m a_m^\dagger a_m, \quad (74)$$

acts on these vectors as follows

$$n|0\rangle = 0, \quad (75)$$

$$n|j\rangle = |j\rangle, \quad (76)$$

$$n|jk\rangle = 2|jk\rangle, \quad (77)$$

$$n|jkl\rangle = 3|jkl\rangle, \quad (78)$$

so it simply counts how many times one acted with creation operators on $|0\rangle$. For this reason n is termed the number-of-particles operator. The subspace consisting of vectors satisfying $n|\psi\rangle = |\psi\rangle$ is called the one-particle sector of the (bosonic) Fock space \mathcal{F} . This is the subspace of \mathcal{F} that is “identical” to the initial Hilbert space \mathcal{H} we have started with. \mathcal{F} is a countable direct sum of n -particle sectors, each of which is in itself a separable Hilbert space. Accordingly, \mathcal{F} is also a separable Hilbert space that could become a starting point for yet another Fock space — and so on and so forth.

In 1925 Heisenberg and his coworkers did not know the notion of a Fock space when they wrote the first paper on field quantization. In fact, they did not even know quantum mechanics. They did not hear of Hilbert spaces or Schrödinger’s equation. But they formulated cornerstones of modern quantum field theory, with all its advantages and defects.

Digression: It is instructive to consider also the following construction. Let us denote by \mathcal{H}_n the Hilbert space spanned by tensor products

$$|j_1, \dots, j_n\rangle = |j_1\rangle \otimes \dots \otimes |j_n\rangle, \quad (79)$$

and let A_n and S_n be the projectors on subspaces of, respectively, anti-symmetric and symmetric states in \mathcal{H}_n (for $n = 0$ we take $\mathcal{H}_0 = \mathbb{C}$; $A_0 = S_0$ and $A_1 = S_1$ are identity operators in \mathcal{H}_0 and \mathcal{H}_1 , respectively). Let $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ and $\alpha_j, \alpha_j^\dagger : \mathcal{H} \rightarrow \mathcal{H}$ be defined by

$$\alpha_j^\dagger |j_1, \dots, j_n\rangle = \bar{c}_{n+1} |j, j_1, \dots, j_n\rangle, \quad (80)$$

$$\alpha_j |j_0, j_1, \dots, j_n\rangle = c_{n+1} \delta_{jj_0} |j_1, \dots, j_n\rangle, \quad (81)$$

$$\alpha_j |0\rangle = 0, \quad (82)$$

$$\alpha_j^\dagger |0\rangle = \bar{c}_1 |j\rangle, \quad (83)$$

$$\alpha_j |j_1\rangle = c_1 \delta_{jj_1} |0\rangle. \quad (84)$$

c_{n+1} and \bar{c}_{n+1} are constants to be specified later. Let $A = \bigoplus_{n=0}^{\infty} A_n$, $S = \bigoplus_{n=0}^{\infty} S_n$, and

$$a_j = S \alpha_j S, \quad (85)$$

$$a_j^\dagger = S \alpha_j^\dagger S, \quad (86)$$

$$b_j = A \alpha_j A, \quad (87)$$

$$b_j^\dagger = A \alpha_j^\dagger A. \quad (88)$$

Let us check:

$$\begin{aligned}
a_j a_j^\dagger |0\rangle &= a_j |j\rangle = c_1 |0\rangle, \\
a_j^\dagger a_j |0\rangle &= 0, \\
[a_j, a_j^\dagger] |0\rangle &= c_1 |0\rangle, \\
a_j a_j^\dagger |j_1\rangle &= \bar{c}_2 a_j \frac{1}{2} (|j, j_1\rangle + |j_1, j\rangle) = \bar{c}_2 S \alpha_j \frac{1}{2} (|j, j_1\rangle + |j_1, j\rangle) = \bar{c}_2 c_2 \frac{1}{2} (|j_1\rangle + \delta_{jj_1} |j\rangle) \\
a_j^\dagger a_j |j_1\rangle &= c_1 \delta_{jj_1} a_j^\dagger |0\rangle = c_1 \bar{c}_1 \delta_{jj_1} |j\rangle, \\
[a_j, a_j^\dagger] |j_1\rangle &= \frac{\bar{c}_2 c_2}{2} |j_1\rangle + \frac{\bar{c}_2 c_2}{2} \delta_{jj_1} |j\rangle - c_1 \bar{c}_1 \delta_{jj_1} |j\rangle, \\
a_j a_j^\dagger |j_1, j_2\rangle &= a_j \alpha_j^\dagger \frac{1}{2} (|j_1, j_2\rangle + |j_2, j_1\rangle) = \bar{c}_3 a_j S \frac{1}{2} (|j\rangle |j_1, j_2\rangle + |j\rangle |j_2, j_1\rangle) \\
&= \bar{c}_3 a_j \frac{1}{6} (|j, j_1, j_2\rangle + |j, j_2, j_1\rangle + |j_1, j, j_2\rangle + |j_1, j_2, j\rangle + |j_2, j_1, j\rangle + |j_2, j, j_1\rangle) \\
&= \bar{c}_3 c_3 S \frac{1}{6} (|j_1, j_2\rangle + |j_2, j_1\rangle + \delta_{jj_1} |j, j_2\rangle + \delta_{jj_1} |j_2, j\rangle + \delta_{jj_2} |j_1, j\rangle + \delta_{jj_2} |j, j_1\rangle) \\
&= \bar{c}_3 c_3 \frac{1}{6} (|j_1, j_2\rangle + |j_2, j_1\rangle + \delta_{jj_1} |j, j_2\rangle + \delta_{jj_1} |j_2, j\rangle + \delta_{jj_2} |j_1, j\rangle + \delta_{jj_2} |j, j_1\rangle) \\
a_j^\dagger a_j |j_1, j_2\rangle &= a_j^\dagger \alpha_j \frac{1}{2} (|j_1, j_2\rangle + |j_2, j_1\rangle) = c_2 S \alpha_j^\dagger S \frac{1}{2} (\delta_{jj_1} |j_2\rangle + \delta_{jj_2} |j_1\rangle) \\
&= c_2 \bar{c}_2 S \frac{1}{2} (\delta_{jj_1} |j, j_2\rangle + \delta_{jj_2} |j, j_1\rangle) \\
&= c_2 \bar{c}_2 \frac{1}{4} (\delta_{jj_1} |j, j_2\rangle + \delta_{jj_1} |j_2, j\rangle + \delta_{jj_2} |j, j_1\rangle + \delta_{jj_2} |j_1, j\rangle) \\
[a_j, a_j^\dagger] |j_1, j_2\rangle &= \bar{c}_3 c_3 \frac{1}{6} (|j_1, j_2\rangle + |j_2, j_1\rangle + \delta_{jj_1} |j, j_2\rangle + \delta_{jj_1} |j_2, j\rangle + \delta_{jj_2} |j_1, j\rangle + \delta_{jj_2} |j, j_1\rangle) \\
&\quad - c_2 \bar{c}_2 \frac{1}{4} (\delta_{jj_1} |j, j_2\rangle + \delta_{jj_1} |j_2, j\rangle + \delta_{jj_2} |j, j_1\rangle + \delta_{jj_2} |j_1, j\rangle)
\end{aligned}$$

The formulas get incredibly simplified if we take $c_n = \bar{c}_n = \sqrt{n}$. Then

$$\begin{aligned}
[a_j, a_j^\dagger] |0\rangle &= |0\rangle, \\
[a_j, a_j^\dagger] |j_1\rangle &= |j_1\rangle, \\
[a_j, a_j^\dagger] |j_1, j_2\rangle &= \frac{1}{2} (|j_1, j_2\rangle + |j_2, j_1\rangle) = S |j_1, j_2\rangle.
\end{aligned}$$

Note that since S is a projector then $[a_j, a_j^\dagger] = [a_j, a_j^\dagger] S$, hence

$$[a_j, a_j^\dagger] S |j_1, j_2\rangle = S |j_1, j_2\rangle.$$

Now take any $n > 1$ and let $\{\sigma = (\sigma(1), \dots, \sigma(n))\}$ be the set of all the permutations of the

sequence $(1, \dots, n)$. Then

$$\begin{aligned}
[a_j, a_j^\dagger] |j_1, \dots, j_n\rangle &= a_j \frac{1}{n!} \sum_{\sigma} \alpha_j^\dagger |j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle - a_j^\dagger \frac{1}{n!} \sum_{\sigma} \alpha_j |j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle \\
&= a_j \frac{\sqrt{n+1}}{n!} \sum_{\sigma} |j, j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle - a_j^\dagger \frac{\sqrt{n}}{n!} \sum_{\sigma} \delta_{jj_{\sigma(1)}} |j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle \\
&= S \alpha_j \frac{\sqrt{n+1}}{n!} \sum_{\sigma} S |j, j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle - S \alpha_j^\dagger \frac{\sqrt{n}}{n!} \sum_{\sigma} \delta_{jj_{\sigma(1)}} S |j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle \\
&= S \alpha_j \frac{\sqrt{n+1}}{n!} \frac{1}{n+1} \sum_{\sigma} \left(|j, j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle + \dots + |j_{\sigma(1)}, \dots, j_{\sigma(n)}, j\rangle \right) \\
&\quad - S \alpha_j^\dagger \frac{\sqrt{n}}{n!} \sum_{\sigma} \delta_{jj_{\sigma(1)}} |j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle \\
&= S \frac{1}{n!} \sum_{\sigma} \left(|j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle + \delta_{jj_{\sigma(1)}} |j, j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle + \dots + \delta_{jj_{\sigma(1)}} |j_{\sigma(2)}, \dots, j_{\sigma(n)}, j\rangle \right) \\
&\quad - S \frac{n}{n!} \sum_{\sigma} \delta_{jj_{\sigma(1)}} |j, j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle \\
&= S \frac{1}{n!} \sum_{\sigma} \left(|j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle + \delta_{jj_{\sigma(1)}} |j, j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle + \dots + \delta_{jj_{\sigma(1)}} |j_{\sigma(2)}, \dots, j_{\sigma(n)}, j\rangle \right) \\
&\quad - S \frac{n}{n!} \frac{1}{n} \sum_{\sigma} \delta_{jj_{\sigma(1)}} \left(|j, j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle + \dots + |j_{\sigma(2)}, \dots, j_{\sigma(n)}, j\rangle \right) \\
&= S \frac{1}{n!} \sum_{\sigma} |j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle \\
&= S |j_1, \dots, j_n\rangle
\end{aligned}$$

Finally,

$$[a_j, a_j^\dagger] = S \quad (89)$$

Analogously

$$\begin{aligned}
b_j b_j^\dagger |0\rangle &= b_j |j\rangle = c_1 |0\rangle, \\
b_j^\dagger b_j |0\rangle &= 0, \\
\{b_j, b_j^\dagger\} |0\rangle &= c_1 |0\rangle, \\
b_j b_j^\dagger |j_1\rangle &= \bar{c}_2 b_j \frac{1}{2} (|j, j_1\rangle - |j_1, j\rangle) = \bar{c}_2 A \alpha_j \frac{1}{2} (|j, j_1\rangle - |j_1, j\rangle) = \bar{c}_2 c_2 \frac{1}{2} (|j_1\rangle - \delta_{jj_1} |j\rangle) \\
b_j^\dagger b_j |j_1\rangle &= c_1 \delta_{jj_1} b_j^\dagger |0\rangle = c_1 \bar{c}_1 \delta_{jj_1} |j\rangle, \\
\{b_j, b_j^\dagger\} |j_1\rangle &= \frac{\bar{c}_2 c_2}{2} |j_1\rangle - \frac{\bar{c}_2 c_2}{2} \delta_{jj_1} |j\rangle + c_1 \bar{c}_1 \delta_{jj_1} |j\rangle, \\
b_j b_j^\dagger |j_1, j_2\rangle &= b_j \alpha_j^\dagger \frac{1}{2} (|j_1, j_2\rangle - |j_2, j_1\rangle) = \bar{c}_3 b_j A \frac{1}{2} (|j, j_1, j_2\rangle - |j, j_2, j_1\rangle) \\
&= \bar{c}_3 A \alpha_j \frac{1}{6} (|j, j_1, j_2\rangle - |j, j_2, j_1\rangle - |j_1, j, j_2\rangle + |j_1, j_2, j\rangle - |j_2, j_1, j\rangle + |j_2, j, j_1\rangle) \\
&= \bar{c}_3 c_3 A \frac{1}{6} (|j_1, j_2\rangle - |j_2, j_1\rangle - \delta_{jj_1} |j, j_2\rangle + \delta_{jj_1} |j_2, j\rangle - \delta_{jj_2} |j_1, j\rangle + \delta_{jj_2} |j, j_1\rangle) \\
&= \bar{c}_3 c_3 \frac{1}{6} (|j_1, j_2\rangle - |j_2, j_1\rangle - \delta_{jj_1} |j, j_2\rangle + \delta_{jj_1} |j_2, j\rangle - \delta_{jj_2} |j_1, j\rangle + \delta_{jj_2} |j, j_1\rangle) \\
b_j^\dagger b_j |j_1, j_2\rangle &= a_j^\dagger \alpha_j \frac{1}{2} (|j_1, j_2\rangle - |j_2, j_1\rangle) = c_2 A \alpha_j^\dagger A \frac{1}{2} (\delta_{jj_1} |j_2\rangle - \delta_{jj_2} |j_1\rangle) \\
&= c_2 \bar{c}_2 A \frac{1}{2} (\delta_{jj_1} |j, j_2\rangle - \delta_{jj_2} |j, j_1\rangle) \\
&= c_2 \bar{c}_2 \frac{1}{4} (\delta_{jj_1} |j, j_2\rangle - \delta_{jj_1} |j_2, j\rangle - \delta_{jj_2} |j, j_1\rangle + \delta_{jj_2} |j_1, j\rangle) \\
\{b_j, b_j^\dagger\} |j_1, j_2\rangle &= \bar{c}_3 c_3 \frac{1}{6} (|j_1, j_2\rangle - |j_2, j_1\rangle - \delta_{jj_1} |j, j_2\rangle + \delta_{jj_1} |j_2, j\rangle - \delta_{jj_2} |j_1, j\rangle + \delta_{jj_2} |j, j_1\rangle) \\
&\quad + c_2 \bar{c}_2 \frac{1}{4} (\delta_{jj_1} |j, j_2\rangle - \delta_{jj_1} |j_2, j\rangle - \delta_{jj_2} |j, j_1\rangle + \delta_{jj_2} |j_1, j\rangle)
\end{aligned}$$

As before, choosing $c_n = \bar{c}_n = \sqrt{n}$, we simplify the formulas. Let $\sum'_\sigma = \sum_\sigma \text{sign}(\sigma)$.

$$\begin{aligned}
\{b_j, b_j^\dagger\} |j_1, \dots, j_n\rangle &= b_j \frac{1}{n!} \sum_\sigma ' \alpha_j^\dagger |j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle + b_j^\dagger \frac{1}{n!} \sum_\sigma ' \alpha_j |j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle \\
&= b_j \frac{\sqrt{n+1}}{n!} \sum_\sigma ' |j, j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle + b_j^\dagger \frac{\sqrt{n}}{n!} \sum_\sigma ' \delta_{jj_{\sigma(1)}} |j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle \\
&= A \alpha_j \frac{\sqrt{n+1}}{n!} \sum_\sigma ' A |j, j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle + A \alpha_j^\dagger \frac{\sqrt{n}}{n!} \sum_\sigma ' \delta_{jj_{\sigma(1)}} A |j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle \\
&= A \alpha_j \frac{\sqrt{n+1}}{n!} \frac{1}{n+1} \sum_\sigma ' (|j, j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle + \dots + (-1)^n |j_{\sigma(1)}, \dots, j_{\sigma(n)}, j\rangle) \\
&\quad + A \alpha_j^\dagger \frac{\sqrt{n}}{n!} \sum_\sigma ' \delta_{jj_{\sigma(1)}} |j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle
\end{aligned}$$

$$\begin{aligned}
&= A \frac{1}{n!} \sum_{\sigma} ' \left(|j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle + (-1)^1 \delta_{jj_{\sigma(1)}} |j, j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle + \dots + (-1)^n \delta_{jj_{\sigma(1)}} |j_{\sigma(2)}, \dots, j_{\sigma(n)}, j\rangle \right) \\
&\quad + A \frac{n}{n!} \sum_{\sigma} ' \delta_{jj_{\sigma(1)}} |j, j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle \\
&= A \frac{1}{n!} \sum_{\sigma} ' \left(|j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle + (-1)^1 \delta_{jj_{\sigma(1)}} |j, j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle + \dots + (-1)^n \delta_{jj_{\sigma(1)}} |j_{\sigma(2)}, \dots, j_{\sigma(n)}, j\rangle \right) \\
&\quad + A \frac{n}{n!} \frac{1}{n} \sum_{\sigma} ' \delta_{jj_{\sigma(1)}} \left(|j, j_{\sigma(2)}, \dots, j_{\sigma(n)}\rangle + \dots + (-1)^{n-1} |j_{\sigma(2)}, \dots, j_{\sigma(n)}, j\rangle \right) \\
&= A \frac{1}{n!} \sum_{\sigma} ' |j_{\sigma(1)}, \dots, j_{\sigma(n)}\rangle \\
&= A |j_1, \dots, j_n\rangle
\end{aligned}$$

Finally,

$$\{b_j, b_j^{\dagger}\} = A. \quad (90)$$

In the subspace of symmetric states $\mathcal{S} \subset \mathcal{H}$, $S\mathcal{S} = \mathcal{S}$, the commutator $[a_j, a_j^{\dagger}] = S$ acts as the identity operator,

$$[a_j, a_j^{\dagger}] \mathcal{S} = \mathcal{S}. \quad (91)$$

Similarly, in the subspace of antisymmetric states $\mathcal{A} \subset \mathcal{H}$, $A\mathcal{A} = \mathcal{A}$, the anti-commutator $\{b_j, b_j^{\dagger}\} = A$ acts as the identity operator,

$$\{b_j, b_j^{\dagger}\} \mathcal{A} = \mathcal{A}. \quad (92)$$

The two algebras (canonical commutation and anti-commutation relations) in physical applications will correspond to different-spin representations of $SU(2)$ or $SL(2, \mathbb{C})$ groups. Therefore formulas such as $a_j^{\dagger}|0\rangle = |j\rangle$ and $b_j^{\dagger}|0\rangle = |j\rangle$ should not both generate the same $|j\rangle$. The simplest solution is to start with $|0, 0\rangle = |0\rangle \otimes |0\rangle$,

$$a_j^{\dagger}|0\rangle \otimes |0\rangle = |j\rangle \otimes |0\rangle \in \mathcal{S} \otimes \mathcal{A}, \quad (93)$$

$$b_j^{\dagger}|0\rangle \otimes |0\rangle = |0\rangle \otimes |j\rangle \in \mathcal{S} \otimes \mathcal{A}, \quad (94)$$

and so on. The algebras then read

$$[a_j, a_j^{\dagger}] = S \otimes I, \quad (95)$$

$$\{b_j, b_j^{\dagger}\} = I \otimes A. \quad (96)$$

This type of construction can be found in the Berezin book [8] but, apparently, goes back to the original works of V. Fock. It is interesting that the right-hand sides of the algebras are not given by the identity operator but by projectors on subspaces of totally symmetric or anti-symmetric states. \blacktriangle

IV. ENSEMBLES OF INDEFINITE-FREQUENCY OSCILLATORS

Energies of classical free fields look in Fourier space analogously to those of ensembles of oscillators. This observation was at the roots of the first approach to field quantization, formulated already in 1925 by Heisenberg, Born and Jordan [9]. One should bear in mind that the first paper of Schrödinger, explaining the role of eigenvalues of operators, appeared a few months later [10]. The authors of [9] apparently did not yet understand the idea of quantum superposition (as suggested by Max Jammer in [11]). Having to consider oscillators with different frequencies they basically had no other option but at least one oscillator per frequency. In this respect nothing essential has changed in mainstream quantum field theory since 1925.

I will now show that their basic assumption is not at all natural. To do so, let us consider a simple 2D pendulum in linear approximation. Classically, its ground state would correspond to no oscillation at all. Quantum mechanically it would imply vanishing momentum and fixed position, a possibility excluded by the uncertainty principle. In consequence one finds the ground state oscillation with energy $\frac{1}{2}\hbar\sqrt{g/l} = \frac{1}{2}\hbar\omega(l)$ (l is the pendulum length). It also practically means that the “lowest atom” of the pendulum is described by the center-of-mass wave packet $\Psi(X)$ where $|\Psi(X)|^2$ is a Gaussian.

If the pendulum is suspended at the origin $(X, Y) = (0, 0)$, the length is given by $l = |Y| = -Y$. For a true pendulum the 2D wave function $\Psi(X, Y)$ would have to be smeared out also in the Y direction and, in fact, a more realistic model should employ a nonseparable potential $U(X, Y) = m\omega(Y)^2 X^2/2$, with deep conditional minimum at $Y = -l$. Note that the frequency is no longer a parameter but an eigenvalue $\omega(Y)$, in position representation, of some operator Ω . Similarly to Schrödinger’s cat that exists in superposition of being dead and alive, our pendulum exists in a superposition of all its possible lengths, so that many different frequencies can be associated with a single oscillator. The example is generic — in sufficiently realistic cases the ω s are not just classical parameters but functions of other

observables.

I don't see any reason why at levels as fundamental as those related to field quantization the appropriate ω s and wave vectors \mathbf{k} should be more classical than Ω from the preceding example. I would rather expect the wave vectors and frequencies to be eigenvalues of some operators. In order to understand formal implications of the latter postulate we first have to understand the simplest examples based on nonrelativistic oscillators.

A. Single indefinite-frequency oscillator

The special case $\Omega = \omega \mathbb{I}$, with parameter ω and identity operator \mathbb{I} , is equivalent to the standard harmonic oscillator.

The simplest nontrivial generalization of $\Omega = \omega \mathbb{I}$ occurs if the operator Ω has discrete spectrum and commutes with canonical momentum and position. So let \hat{p} and \hat{x} be the canonical momentum and position acting in the Hilbert space spanned by the number-of-excitations eigenvectors $|n\rangle$. Now consider the following representation

$$\Omega = \sum_{\omega} \omega |\omega\rangle \langle \omega| \otimes I, \quad (97)$$

$$P = \sum_{\omega} |\omega\rangle \langle \omega| \otimes \hat{p} = I_{\Omega} \otimes \hat{p}, \quad (98)$$

$$Q = \sum_{\omega} |\omega\rangle \langle \omega| \otimes \hat{x} = I_{\Omega} \otimes \hat{x}, \quad (99)$$

with Hamiltonian of the usual form

$$H = \frac{P^2}{2m} + \frac{m\Omega^2 Q^2}{2} = \frac{\hbar\Omega}{2} (a_{\Omega}^{\dagger} a_{\Omega} + a_{\Omega} a_{\Omega}^{\dagger}) = \hbar\Omega \left(a_{\Omega}^{\dagger} a_{\Omega} + \frac{1}{2} I_{\Omega} \otimes I \right), \quad (100)$$

and CCR

$$a_{\Omega} = \frac{1}{\sqrt{2\hbar m\Omega}} (m\Omega Q + iP), \quad (101)$$

$$a_{\Omega}^{\dagger} = \frac{1}{\sqrt{2\hbar m\Omega}} (m\Omega Q - iP), \quad (102)$$

acting in the Hilbert space spanned by $|\omega, n\rangle = |\omega\rangle \otimes |n\rangle$. For any ω

$$H|\omega, n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |\omega, n\rangle, \quad (103)$$

$$a_{\Omega}|\omega, 0\rangle = 0. \quad (104)$$

Let

$$|\psi\rangle = \sum_{\omega, n} \psi_{\omega, n} |\omega, n\rangle \quad (105)$$

be an arbitrary state. The average

$$\langle\psi|H|\psi\rangle = \sum_{\omega, n} \hbar\omega \left(n + \frac{1}{2}\right) |\psi_{\omega, n}|^2 \quad (106)$$

looks as an average energy of an ensemble of harmonic oscillators, different frequencies occurring with probabilities

$$p_\omega = \sum_{n=0}^{\infty} |\psi_{\omega, n}|^2. \quad (107)$$

We can also write

$$\begin{aligned} \langle\psi|H|\psi\rangle &= \sum_{\omega} \hbar\omega \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) |\psi_{\omega, n}|^2 \\ &= \sum_{\omega} \frac{\hbar\omega}{2} \langle\psi| \left(|\omega\rangle\langle\omega| \otimes (\hat{a}_{\omega}^{\dagger}\hat{a}_{\omega} + \hat{a}_{\omega}\hat{a}_{\omega}^{\dagger})\right) |\psi\rangle. \\ &= \sum_{\omega} \frac{\hbar\omega}{2} \langle\psi| (a_{\omega}^{\dagger}a_{\omega} + a_{\omega}a_{\omega}^{\dagger}) |\psi\rangle. \end{aligned} \quad (108)$$

Here

$$\hat{a}_{\omega} = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega\hat{x} + i\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}, \quad (109)$$

$$\hat{a}_{\omega}^{\dagger} = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega\hat{x} - i\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2\hbar m\omega}}\hat{p}, \quad (110)$$

$$\hat{x} = \sqrt{\frac{2\hbar}{m\omega}} (\hat{a}_{\omega} + \hat{a}_{\omega}^{\dagger}), \quad (111)$$

are the usual creation and annihilation operators, and

$$a_{\omega} = |\omega\rangle\langle\omega| \otimes \hat{a}_{\omega}, \quad (112)$$

$$a_{\omega}^{\dagger} = |\omega\rangle\langle\omega| \otimes \hat{a}_{\omega}^{\dagger}, \quad (113)$$

$$|\omega\rangle\langle\omega| \otimes \hat{x} = |\omega\rangle\langle\omega| \otimes \sqrt{\frac{2\hbar}{m\omega}} (\hat{a}_{\omega} + \hat{a}_{\omega}^{\dagger}) \quad (114)$$

$$= \sqrt{\frac{2\hbar}{m\omega}} (a_{\omega} + a_{\omega}^{\dagger}) = Q_{\omega}. \quad (115)$$

We have arrived at the decomposition

$$H = \sum_{\omega} H_{\omega}, \quad (116)$$

$$H_{\omega} = \frac{\hbar\omega}{2}(a_{\omega}^{\dagger}a_{\omega} + a_{\omega}a_{\omega}^{\dagger}) \quad (117)$$

$$= \hbar\omega n_{\omega} + \frac{\hbar\omega}{2}I_{\omega} \quad (118)$$

$$\langle\psi|H_{\omega}|\psi\rangle = \hbar\omega \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) |\psi_{\omega,n}|^2. \quad (119)$$

As we can see, H_{ω} has properties of Hamiltonian of an oscillator whose frequency is ω , and H is a sum of such Hamiltonians taken over all the eigenvalues of Ω . The new feature is the fact that H describes a single harmonic oscillator existing in superposition of different ω s.

In Heisenberg picture we find

$$a_{\omega}(t) = e^{iHt/\hbar}a_{\omega}e^{-iHt/\hbar} = a_{\omega}e^{-i\omega t}, \quad (120)$$

$$a_{\omega}(t)^{\dagger} = e^{iHt/\hbar}a_{\omega}^{\dagger}e^{-iHt/\hbar} = a_{\omega}^{\dagger}e^{i\omega t}, \quad (121)$$

$$\hat{x}(t) = \sqrt{\frac{2\hbar}{m\omega}}\left(\hat{a}_{\omega}e^{-i\omega t} + \hat{a}_{\omega}^{\dagger}e^{i\omega t}\right), \quad (122)$$

$$Q_{\omega}(t) = \sqrt{\frac{2\hbar}{m\omega}}\left(a_{\omega}e^{-i\omega t} + a_{\omega}^{\dagger}e^{i\omega t}\right), \quad (123)$$

which are again the standard formulas. However, in spite of all these similarities to the known textbook results, the representation of HOLA is here a nonstandard one:

$$n_{\omega} = a_{\omega}^{\dagger}a_{\omega} = |\omega\rangle\langle\omega| \otimes \hat{a}_{\omega}^{\dagger}\hat{a}_{\omega}, \quad (124)$$

$$I_{\omega} = |\omega\rangle\langle\omega| \otimes I, \quad (125)$$

$$[a_{\omega}, a_{\omega'}^{\dagger}] = \delta_{\omega,\omega'}I_{\omega}, \quad (126)$$

$$[a_{\omega}, n_{\omega'}] = \delta_{\omega,\omega'}a_{\omega}, \quad (127)$$

$$[a_{\omega}^{\dagger}, n_{\omega'}] = -\delta_{\omega,\omega'}a_{\omega}^{\dagger}. \quad (128)$$

I_{ω} commutes with all the elements of the Lie algebra, but is not proportional to the identity. By Schur's lemma the representation is thus reducible. Its irreducible representation components are spanned by $|\omega, n\rangle$, with fixed ω and arbitrary n . The projector on this subspace is given by I_{ω} itself.

Physically, the above representation describes a single oscillator whose states are wave packets consisting of superpositions of different eigenvalues ω .

Remark: Let us note that (109) and (110) imply

$$\begin{aligned}
[\hat{a}_\omega, \hat{a}_{\omega'}^\dagger] &= \left[\frac{1}{\sqrt{2\hbar m\omega}} (m\omega\hat{x} + i\hat{p}), \frac{1}{\sqrt{2\hbar m\omega'}} (m\omega'\hat{x} - i\hat{p}) \right] \\
&= \frac{1}{2\hbar m\sqrt{\omega\omega'}} [m\omega\hat{x}, -i\hat{p}] + \frac{1}{2\hbar m\sqrt{\omega\omega'}} [i\hat{p}, m\omega'\hat{x}] \\
&= \frac{\omega + \omega'}{2\sqrt{\omega\omega'}} = \iota(\omega, \omega').
\end{aligned} \tag{129}$$

The right side of (129) is a function satisfying

$$\iota(\omega, \omega') = \iota(\omega', \omega), \tag{130}$$

$$\iota(\omega, \omega) = 1, \tag{131}$$

which is neither 1 nor any kind of delta. The explicit form of $\iota(\omega, \omega')$ does not occur in (126) due to the presence of $|\omega\rangle\langle\omega|$ in (112) and (113). We should keep this observation in mind when we perform field quantization.▲

B. Several indefinite-frequency oscillators

Now, what about two such oscillators? One can trivially extend all the operators by

$$a_\omega \rightarrow a_\omega \otimes I(1), \tag{132}$$

$$a_\omega \rightarrow I(1) \otimes a_\omega, \tag{133}$$

and so on, where

$$I(1) = \sum_\omega I_\omega \tag{134}$$

is the identity in the one-oscillator Hilbert space spanned by $|\omega, n\rangle$. However, if one additionally requires their bosonic statistics, something one expects for spin-0 fields, say, the operators should preserve symmetry of states. The natural bosonic generalization is then

$$a_\omega(2) = c_2 \left(a_\omega \otimes I(1) + I(1) \otimes a_\omega \right) \tag{135}$$

where c_2 is a constant, and

$$\begin{aligned}
[a_\omega(2), a_{\omega'}(2)^\dagger] &= \delta_{\omega, \omega'} |c_2|^2 \left(I_\omega \otimes I(1) + I(1) \otimes I_\omega \right), \\
&= \delta_{\omega, \omega'} I_\omega(2).
\end{aligned} \tag{136}$$

Let us note that

$$\sum_{\omega} I_{\omega}(2) = 2|c_2|^2 I(1) \otimes I(1) = 2|c_2|^2 I(2) \quad (137)$$

suggesting the normalization $c_2 = 1/\sqrt{2}$, analogous to (8). In order to get the whole HOLA we have to define $n_{\omega}(2)$. The first guess

$$n_{\omega}(2) = a_{\omega}(2)^{\dagger} a_{\omega}(2) \quad (138)$$

leads to

$$[a_{\omega}(2), n_{\omega}(2)] = \delta_{\omega, \omega'} I_{\omega}(2) a_{\omega}(2) \quad (139)$$

which is not even a Lie algebra. (Certain observables do satisfy such generalized Lie algebras — Hamiltonian, angular momentum, and the Runge-Lenz vector of the Coulomb problem provide an example.) But we remember that in standard representations of CCR we have encountered the ambiguity of (11) versus (18). In the present context the second option reads

$$\tilde{n}_{\omega}(2) = n_{\omega} \otimes I(1) + I(1) \otimes n_{\omega}, \quad (140)$$

leading to HOLA

$$[a_{\omega}(2), a_{\omega'}(2)^{\dagger}] = \delta_{\omega, \omega'} I_{\omega}(2), \quad (141)$$

$$[a_{\omega}(2), \tilde{n}_{\omega'}(2)] = \delta_{\omega, \omega'} a_{\omega}(2), \quad (142)$$

$$[a_{\omega}(2)^{\dagger}, \tilde{n}_{\omega'}(2)] = -\delta_{\omega, \omega'} a_{\omega}(2)^{\dagger}. \quad (143)$$

This representation is again reducible since

$$I_{\omega}(2) = \frac{1}{2} (I_{\omega} \otimes I(1) + I(1) \otimes I_{\omega}) \neq I(2), \quad (144)$$

$$\sum_{\omega} I_{\omega}(2) = I(2). \quad (145)$$

$\tilde{n}_{\omega}(2)$ counts the number of excitations of the two-oscillator system. Especially interesting is the form of $I_{\omega}(2)$: This is the number-of-successes (in two trials) operator known from works on probabilistic interpretation of quantum mechanics [12–16]. Eigenvalues of $I_{\omega}(2)$ are 0, 1/2, 1 and describe the fraction of positive answers (in two trials) to the question “is

the frequency of the oscillator equal to ω ?", if ω is selected at random. Due to the resolution of identity (145) the element $I_\omega(2)$ is a positive operator-valued measure [17].

An extension to arbitrary number N of oscillators is now clear, with $c_N = 1/\sqrt{N}$. The fact that $I_\omega(N)$ becomes the N -trial number-of-successes operator is essential for the limit $N \rightarrow \infty$ which can (and later will) be treated by weak laws of large numbers.

Before I make a digression on scalar fields let me note that in Heisenberg picture, for arbitrary $N \geq 1$, we find

$$a_\omega(t, N) = e^{i\tilde{H}(N)t/\hbar} a_\omega(N) e^{-i\tilde{H}(N)t/\hbar} = a_\omega(N) e^{-i\omega t}, \quad (146)$$

if the Hamiltonian is given by $\tilde{H}(N) = \sum_\omega \hbar\omega \tilde{n}_\omega(N)$. For $H(N) = \sum_\omega \hbar\omega a_\omega(N)^\dagger a_\omega(N)$ we would get

$$a_\omega(t, N) = e^{iH(N)t/\hbar} a_\omega(N) e^{-iH(N)t/\hbar} = a_\omega(N) e^{-i\omega I_\omega(N)t}. \quad (147)$$

This is a strong argument in favor of $\tilde{H}(N)$ in contrast to $H(N)$... (Still, as mentioned already in [7], I have doubts here — the choice of $H(N)$ is appealing for various reasons, so we will devote some time also to this option.)

Canonical position operators

$$Q_\omega(t, N) = \sqrt{\frac{2\hbar}{m\omega}} \left(a_\omega(N) e^{-i\omega t} + a_\omega(N)^\dagger e^{i\omega t} \right), \quad (148)$$

$$Q(t, N) = \sum_\omega \sqrt{\frac{2\hbar}{m\omega}} \left(a_\omega(N) e^{-i\omega t} + a_\omega(N)^\dagger e^{i\omega t} \right), \quad (149)$$

are formally very similar to scalar-field operators. This observation is behind the basic physical intuition that leads to quantum fields, quantized in “my” way.

V. FREQUENCY-OF-SUCCESSSES OPERATOR AND HOLA

For $N = 1$ the operators $\Pi_\omega^{(1)} = I_\omega$ and $\Pi_\omega^{(0)} = I(1) - I_\omega$ are orthogonal projectors : $(\Pi_\omega^{(0)})^2 = \Pi_\omega^{(0)}$, $(\Pi_\omega^{(1)})^2 = \Pi_\omega^{(1)}$, $\Pi_\omega^{(0)}\Pi_\omega^{(1)} = 0$, $\Pi_\omega^{(0)} + \Pi_\omega^{(1)} = I(1)$. The central element

$$I_\omega(N) = \frac{1}{N} \left(I_\omega \otimes I(1) \otimes \cdots \otimes I(1) + \cdots + I(1) \otimes \cdots \otimes I(1) \otimes I_\omega \right) \quad (150)$$

has eigenvalues $0/N, 1/N, 2/N, \dots, (N-1)/N, N/N$. Its spectral decomposition can be deduced from

$$I(1) = \Pi_{\omega}^{(0)} + \Pi_{\omega}^{(1)} = \sum_{s=0}^1 \Pi_{\omega}^{(s)}, \quad (151)$$

$$\begin{aligned} I_{\omega}(N) &= \frac{1}{N} \left(\Pi_{\omega}^{(1)} \otimes I(1) \otimes \dots \otimes I(1) + \dots + I(1) \otimes \dots \otimes I(1) \otimes \Pi_{\omega}^{(1)} \right) \\ &= \frac{0}{N} \Pi_{\omega}^{(0)} \otimes \dots \otimes \Pi_{\omega}^{(0)} \\ &\quad + \frac{1}{N} \left(\Pi_{\omega}^{(1)} \otimes I(1) \otimes \dots \otimes I(1) + \dots + I(1) \otimes \dots \otimes I(1) \otimes \Pi_{\omega}^{(1)} \right) \end{aligned} \quad (152)$$

and

$$I_{\omega}(N) \Pi_{\omega}^{(s_1)} \otimes \dots \otimes \Pi_{\omega}^{(s_N)} = \frac{s_1 + \dots + s_N}{N} \Pi_{\omega}^{(s_1)} \otimes \dots \otimes \Pi_{\omega}^{(s_N)}. \quad (153)$$

Since

$$\sum_{s_1, \dots, s_N=0}^1 \Pi_{\omega}^{(s_1)} \otimes \dots \otimes \Pi_{\omega}^{(s_N)} = I(N) \quad (154)$$

we arrive at

$$\begin{aligned} I_{\omega}(N) &= \sum_{s_1, \dots, s_N=0}^1 I_{\omega}(N) \Pi_{\omega}^{(s_1)} \otimes \dots \otimes \Pi_{\omega}^{(s_N)} \\ &= \sum_{s_1, \dots, s_N=0}^1 \frac{s_1 + \dots + s_N}{N} \Pi_{\omega}^{(s_1)} \otimes \dots \otimes \Pi_{\omega}^{(s_N)} = \sum_{s=0}^N \frac{s}{N} \Pi_{\omega} \left(\frac{s}{N} \right), \end{aligned} \quad (155)$$

$$\Pi_{\omega} \left(\frac{s}{N} \right) = \sum_{s_1 + \dots + s_N = s} \Pi_{\omega}^{(s_1)} \otimes \dots \otimes \Pi_{\omega}^{(s_N)}. \quad (156)$$

Now let $|\psi, 1\rangle = \sum_{\omega, n} \psi_{\omega, n} |\omega, n\rangle$, $|\psi, N\rangle = |\psi, 1\rangle \otimes \dots \otimes |\psi, 1\rangle$. The average

$$\langle \psi, 1 | I_{\omega} | \psi, 1 \rangle = \langle \psi, 1 | \Pi_{\omega}^{(1)} | \psi, 1 \rangle = \sum_n |\psi_{\omega, n}|^2 = p_{\omega}, \quad (157)$$

is the probability of finding ω . The average

$$\begin{aligned} \langle \psi, N | \Pi_{\omega} \left(\frac{s}{N} \right) | \psi, N \rangle &= \sum_{s_1 + \dots + s_N = s} \langle \psi, N | \Pi_{\omega}^{(s_1)} \otimes \dots \otimes \Pi_{\omega}^{(s_N)} | \psi, N \rangle \\ &= \sum_{s_1 + \dots + s_N = s} \langle \psi, 1 | \Pi_{\omega}^{(s_1)} | \psi, 1 \rangle \dots \langle \psi, 1 | \Pi_{\omega}^{(s_N)} | \psi, 1 \rangle \\ &= \binom{N}{s} p_{\omega}^s (1 - p_{\omega})^{N-s} \end{aligned} \quad (158)$$

is the probability of finding ω exactly s times in N measurements, performed on each of the N oscillators once, if each oscillator is in state $|\psi, 1\rangle$.

If $F : [0, 1] \rightarrow \mathbb{R}$ is continuous, then by the weak law of large numbers for binomial distribution (or, more generally, Feller's theorem [18])

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle \psi, N | F(I_\omega(N)) | \psi, N \rangle &= \lim_{N \rightarrow \infty} \langle \psi, N | \sum_{s=0}^N F\left(\frac{s}{N}\right) \Pi_\omega\left(\frac{s}{N}\right) | \psi, N \rangle \\ &= \lim_{N \rightarrow \infty} \sum_{s=0}^N F\left(\frac{s}{N}\right) \binom{N}{s} p_\omega^s (1 - p_\omega)^{N-s} \\ &= F(p_\omega). \end{aligned} \tag{159}$$

So, in practice, predictions concerning the gas consisting of N indefinite-frequency oscillators are in the limit $N \rightarrow \infty$ close to those of the standard oscillators with the modified CCR $[a_\omega, a_{\omega'}^\dagger] = p_\omega \delta_{\omega\omega'}$. This result will be the basis of the correspondence principle relating standard regularized quantum field theory with the one I propose in these notes. In fact, as discussed in detail in [19–21], there exist two physically meaningful limits $N \rightarrow \infty$. One is just the weak law of large numbers while the second one is a kind of thermodynamic limit.

VI. STATES OF INDEFINITE-FREQUENCY OSCILLATORS

States of indefinite-frequency oscillators have properties that will be later used in construction of vacuum, n -particle and coherent states of quantum fields.

A. N -oscillator analogues of n -photon states

The “vacuum subspace” consists of vectors that are annihilated by all annihilation operators. The N -oscillator vacuum subspace is spanned by $|\omega_1, 0\rangle \otimes \cdots \otimes |\omega_N, 0\rangle$. I will assume that vacua are given by pure product states

$$|O, N\rangle = \sum_{\omega_1, \dots, \omega_N} O_{\omega_1} \cdots O_{\omega_N} |\omega_1, 0\rangle \otimes \cdots \otimes |\omega_N, 0\rangle, \tag{160}$$

where the O_ω s play a role of a single-oscillator wave function, normalized by

$$\sum_{\omega} |O_\omega|^2 = 1. \tag{161}$$

$p_\omega = |O_\omega|^2$ is the probability that a given oscillator has frequency ω .

Let us begin with the analog of an ordinary 1-photon state, that is $a_\omega(N)^\dagger|O, N\rangle$. Its squared norm is

$$\begin{aligned}\langle O, N|a_\omega(N)a_\omega(N)^\dagger|O, N\rangle &= \langle O, N|I_\omega(N)|O, N\rangle \\ &= \langle O, 1|I_\omega|O, 1\rangle \\ &= |O_\omega|^2 = p_\omega.\end{aligned}\tag{162}$$

Whatever representation of $n_\omega(N)$ satisfying HOLA $[n_\omega(N), a_{\omega'}(N)^\dagger] = \delta_{\omega\omega'}a_{\omega'}(N)^\dagger$ and annihilating $|O, N\rangle$ we take, we find that $a_\omega(N)^\dagger|O, N\rangle$ is a “single-excitation” state, i.e.

$$n_\omega(N)a_\omega(N)^\dagger|O, N\rangle = [n_\omega(N), a_\omega(N)^\dagger]|O, N\rangle = a_\omega(N)^\dagger|O, N\rangle.\tag{163}$$

The peculiarity of this representation of HOLA is that there exist “single-excitation” states that are not spanned by vectors $a_\omega(N)^\dagger|O, N\rangle$. A simple (and generic) example is provided by any state of the form

$$F(I_\omega(N))a_\omega(N)^\dagger|O, N\rangle.\tag{164}$$

In general, n -excitation states are any states spanned by

$$|\omega_1, n_1\rangle \otimes \cdots \otimes |\omega_N, n_N\rangle, \quad n_1 + \cdots + n_N = n.\tag{165}$$

Their particular sub-class is given by

$$a_{\omega_1}(N)^\dagger \cdots a_{\omega_n}(N)^\dagger|O, N\rangle.\tag{166}$$

Indeed, each action of a creation operator adds one excitation to the state, and $|O, N\rangle$ has zero excitations.

B. N -oscillator analogues of coherent states

Quantum mechanics textbooks give (at least) two definitions of coherent states of a harmonic oscillator: States generated by displacement operators acting on the oscillator ground state and eigenstates of annihilation operators. In reducible representations of HOLA the two definitions are not equivalent.

The displacement operator

$$\begin{aligned}
\mathcal{D}(\alpha, N) &= \exp \left(\sum_{\omega} \alpha_{\omega} a_{\omega}(N)^{\dagger} - \sum_{\omega} \overline{\alpha_{\omega}} a_{\omega}(N) \right) \\
&= \exp \left(-\frac{1}{2} \sum_{\omega} |\alpha_{\omega}|^2 I_{\omega}(N) \right) \exp \left(\sum_{\omega} \alpha_{\omega} a_{\omega}(N)^{\dagger} \right) \exp \left(- \sum_{\omega} \overline{\alpha_{\omega}} a_{\omega}(N) \right).
\end{aligned} \tag{167}$$

is unitary

$$\mathcal{D}(\alpha, N)^{\dagger} = \mathcal{D}(\alpha, N)^{-1} = \mathcal{D}(-\alpha, N). \tag{168}$$

The operator can be also represented in the following useful way

$$\begin{aligned}
\mathcal{D}(\alpha, N) &= e^{\frac{1}{\sqrt{N}} \sum_{\omega} (\alpha_{\omega} a_{\omega}(1)^{\dagger} - \overline{\alpha_{\omega}} a_{\omega}(1))} \otimes \dots \otimes e^{\frac{1}{\sqrt{N}} \sum_{\omega} (\alpha_{\omega} a_{\omega}(1)^{\dagger} - \overline{\alpha_{\omega}} a_{\omega}(1))} \\
&= \mathcal{D}(\alpha/\sqrt{N}, 1) \otimes \dots \otimes \mathcal{D}(\alpha/\sqrt{N}, 1) \\
&= \mathcal{D}(\alpha/\sqrt{N}, 1)^{\otimes N},
\end{aligned} \tag{169}$$

$$\begin{aligned}
\mathcal{D}(\alpha, 1) &= \exp \left(\sum_{\omega} |\omega\rangle \langle \omega| \otimes (\alpha_{\omega} \hat{a}^{\dagger} - \overline{\alpha_{\omega}} \hat{a}) \right) \\
&= \sum_{\omega} |\omega\rangle \langle \omega| \otimes \exp (\alpha_{\omega} \hat{a}^{\dagger} - \overline{\alpha_{\omega}} \hat{a}) \\
&= \sum_{\omega} |\omega\rangle \langle \omega| \otimes \exp (-|\alpha_{\omega}|^2/2) \exp (\alpha_{\omega} \hat{a}^{\dagger}) \exp (-\overline{\alpha_{\omega}} \hat{a}) \\
&= \sum_{\omega} |\omega\rangle \langle \omega| \otimes \hat{\mathcal{D}}(\alpha_{\omega}).
\end{aligned} \tag{170}$$

$\hat{\mathcal{D}}(\alpha_{\omega})$ is the ordinary displacement operator known from quantum mechanics textbooks.

The name of the operator comes from the following property

$$\mathcal{D}(\alpha, N)^{\dagger} a_{\omega}(N) \mathcal{D}(\alpha, N) = a_{\omega}(N) + \alpha_{\omega} I_{\omega}(N), \tag{171}$$

$$\mathcal{D}(\alpha, N)^{\dagger} a_{\omega}(N)^{\dagger} \mathcal{D}(\alpha, N) = a_{\omega}(N)^{\dagger} + \overline{\alpha_{\omega}} I_{\omega}(N), \tag{172}$$

$$\mathcal{D}(\alpha, N)^{\dagger} I_{\omega}(N) \mathcal{D}(\alpha, N) = I_{\omega}(N). \tag{173}$$

Combining these formulas we obtain a generalized eigenvalue problem

$$\begin{aligned}
a_{\omega}(N) \mathcal{D}(\alpha, N) |O, N\rangle &= \mathcal{D}(\alpha, N) \mathcal{D}(\alpha, N)^{\dagger} a_{\omega}(N) \mathcal{D}(\alpha, N) |O, N\rangle \\
&= \mathcal{D}(\alpha, N) \left(a_{\omega}(N) + \alpha_{\omega} I_{\omega}(N) \right) |O, N\rangle \\
&= \alpha_{\omega} \mathcal{D}(\alpha, N) I_{\omega}(N) |O, N\rangle \\
&= \alpha_{\omega} I_{\omega}(N) \mathcal{D}(\alpha, N) |O, N\rangle.
\end{aligned} \tag{174}$$

The coherent state

$$|\alpha, N\rangle = \mathcal{D}(\alpha, N)|O, N\rangle \quad (175)$$

$$= \exp\left(-\frac{1}{2}\sum_{\omega}|\alpha_{\omega}|^2 I_{\omega}(N)\right) \exp\left(\sum_{\omega}\alpha_{\omega}a_{\omega}(N)^{\dagger}\right)|O, N\rangle \quad (176)$$

$$= \mathcal{D}(\alpha/\sqrt{N}, 1)|O, 1\rangle \otimes \cdots \otimes \mathcal{D}(\alpha/\sqrt{N}, 1)|O, 1\rangle \quad (177)$$

$$= |\alpha/\sqrt{N}, 1\rangle \otimes \cdots \otimes |\alpha/\sqrt{N}, 1\rangle \quad (178)$$

satisfies

$$a_{\omega}(N)|\alpha, N\rangle = \alpha_{\omega}I_{\omega}(N)|\alpha, N\rangle. \quad (179)$$

The latter is a generalized eigenvalue equation in the sense that it combines several ordinary eigenvalue problems that can be extracted from (179) by means of $\Pi_{\omega}(s/N)$:

$$\begin{aligned} a_{\omega}(N)\Pi_{\omega}(s/N)|\alpha, N\rangle &= \Pi_{\omega}(s/N)a_{\omega}(N)|\alpha, N\rangle \\ &= \Pi_{\omega}(s/N)\alpha_{\omega}I_{\omega}(N)|\alpha, N\rangle \\ &= \frac{s}{N}\alpha_{\omega}\Pi_{\omega}(s/N)|\alpha, N\rangle. \end{aligned} \quad (180)$$

Vector $\Pi_{\omega}(s/N)|\alpha, N\rangle$ is the eigenvector of $a_{\omega}(N)$ with the eigenvalue $s\alpha_{\omega}/N$.

The difference with respect to the “ordinary” harmonic oscillator known from textbooks is that the latter involves $I_{\omega} = 1$ so that the coherent state is just a single eigenvector with eigenvalue α_{ω} . We will later see that the presence of s/N may make various quantum field theoretical predictions more physical (if quantum fields are defined by means of an analogous representation of HOLA).

C. Statistics of excitations

The n -excitation subspace is spanned by vectors (165). The projector on this subspace is given by

$$\begin{aligned} \Pi(n, N) &= \sum_{n_1+\cdots+n_N=n} \sum_{\omega_1\ldots\omega_N} |\omega_1, n_1\rangle\langle\omega_1, n_1| \otimes \cdots \otimes |\omega_N, n_N\rangle\langle\omega_N, n_N| \\ &= \sum_{n_1+\cdots+n_N=n} \left(I_{\Omega} \otimes |n_1\rangle\langle n_1|\right) \otimes \cdots \otimes \left(I_{\Omega} \otimes |n_N\rangle\langle n_N|\right). \end{aligned} \quad (181)$$

Probability of finding n excitations in a 1-oscillator coherent state is

$$p(n, 1) = \langle\alpha, 1|\Pi(n, 1)|\alpha, 1\rangle = \sum_{\omega} |O_{\omega}|^2 \frac{(|\alpha_{\omega}|^2)^n}{n!} e^{-\sum_{\omega} |\alpha_{\omega}|^2}. \quad (182)$$

(182) is the Poisson distribution typical of standard coherent states weighted by probability that frequency of the oscillator equals ω . This is precisely the result one expected for a single oscillator wave packet.

Probability of finding n excitations in an N -oscillator coherent state can be computed if one recalls that (166) belongs to the n -excitation subspace. Therefore,

$$\begin{aligned} p(n, N) &= \langle \alpha, N | \Pi(n, N) | \alpha, N \rangle \\ &= \frac{1}{(n!)^2} \langle O, N | \exp \left(- \sum_{\omega} |\alpha_{\omega}|^2 I_{\omega}(N) \right) \left(\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \right)^n \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^n | O, N \rangle \end{aligned}$$

Taking into account

$$\begin{aligned} \left[\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N), \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^n \right] &= \sum_{\omega_1} \overline{\alpha_{\omega_1}} \left[a_{\omega_1}(N), \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^n \right] \\ &= n \sum_{\omega_1} |\alpha_{\omega_1}|^2 I_{\omega_1}(N) \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^{n-1} \end{aligned}$$

and acting with

$$\begin{aligned} &\left[\left(\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \right)^n, \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^n \right] \\ &= \left(\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \right)^{n-1} \left[\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N), \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^n \right] \\ &\quad + \left[\left(\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \right)^{n-1}, \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^n \right] \sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \\ &= n \sum_{\omega_1} |\alpha_{\omega_1}|^2 I_{\omega_1}(N) \left(\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \right)^{n-1} \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^{n-1} \\ &\quad + \left[\left(\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \right)^{n-1}, \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^n \right] \sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \end{aligned} \quad (183)$$

on $|O, N\rangle$, one finds

$$\begin{aligned} &\left(\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \right)^n \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^n |O, N\rangle \\ &= \left[\left(\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \right)^n, \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^n \right] |O, N\rangle \\ &= \left(\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \right)^{n-1} \left[\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N), \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^n \right] |O, N\rangle \\ &= n \sum_{\omega_1} |\alpha_{\omega_1}|^2 I_{\omega_1}(N) \left(\sum_{\omega_1} \overline{\alpha_{\omega_1}} a_{\omega_1}(N) \right)^{n-1} \left(\sum_{\omega_2} \alpha_{\omega_2} a_{\omega_2}(N)^{\dagger} \right)^{n-1} |O, N\rangle \\ &= n! \left(\sum_{\omega_1} |\alpha_{\omega_1}|^2 I_{\omega_1}(N) \right)^n |O, N\rangle. \end{aligned}$$

The end result

$$\begin{aligned}
p(n, N) &= \langle O, N | \exp \left(- \sum_{\omega} |\alpha_{\omega}|^2 I_{\omega}(N) \right) \frac{1}{n!} \left(\sum_{\omega_1} |\alpha_{\omega_1}|^2 I_{\omega_1}(N) \right)^n | O, N \rangle \\
&= \frac{1}{n!} \frac{d^n}{d\lambda^n} \langle O, N | \exp \left(\lambda \sum_{\omega} |\alpha_{\omega}|^2 I_{\omega}(N) \right) | O, N \rangle \Big|_{\lambda=-1}.
\end{aligned} \tag{184}$$

is a generalized Poisson distribution. In order to better understand the generalization we have found let us take a closer look at the generating function

$$\begin{aligned}
\langle O, N | \exp \left(\lambda \sum_{\omega} |\alpha_{\omega}|^2 I_{\omega}(N) \right) | O, N \rangle &= \langle O, 1 | \exp \left(\lambda \sum_{\omega} \frac{1}{N} |\alpha_{\omega}|^2 I_{\omega} \right) | O, 1 \rangle^N \\
&= \langle O, 1 | \sum_{\omega} I_{\omega} \exp \left(\frac{1}{N} \lambda |\alpha_{\omega}|^2 \right) | O, 1 \rangle^N \\
&= \left(\sum_{\omega} |O_{\omega}|^2 e^{\lambda \frac{1}{N} |\alpha_{\omega}|^2} \right)^N.
\end{aligned}$$

The possibility of taking the sum in front of the exponent comes from the fact that $I_{\omega} = |\omega\rangle\langle\omega| \otimes 1$ is a projector. Now, let us introduce a new parameter q satisfying $1 - q = 1/N$,

$$\begin{aligned}
\left(\sum_{\omega} |O_{\omega}|^2 e^{(1-q)\lambda |\alpha_{\omega}|^2} \right)^{\frac{1}{1-q}} &= \exp \ln \left(\sum_{\omega} |O_{\omega}|^2 e^{(1-q)\lambda |\alpha_{\omega}|^2} \right)^{\frac{1}{1-q}} \\
&= \exp \left(\frac{1}{1-q} \ln \left(\sum_{\omega} |O_{\omega}|^2 e^{(1-q)\lambda |\alpha_{\omega}|^2} \right) \right).
\end{aligned} \tag{185}$$

Expression under the exponent,

$$\frac{1}{1-q} \ln \left(\sum_{\omega} |O_{\omega}|^2 e^{(1-q)\lambda |\alpha_{\omega}|^2} \right) \tag{186}$$

is well known in probability and information theory: This is the Kolmogorov-Nagumo average of the random variable $\lambda |\alpha_{\omega}|^2$ [22], introduced by Alfréd Rényi in his derivation of generalized entropies [23]. Since

$$\lim_{q \rightarrow 1} \ln \left(\sum_{\omega} |O_{\omega}|^2 e^{(1-q)\lambda |\alpha_{\omega}|^2} \right) = \ln \left(\underbrace{\sum_{\omega} |O_{\omega}|^2}_{\langle O, 1 | O, 1 \rangle = 1} \right) = 0, \tag{187}$$

we can use the de l'Hospital rule to compute the limit

$$\lim_{q \rightarrow 1} \frac{1}{1-q} \ln \left(\sum_{\omega} |O_{\omega}|^2 e^{(1-q)\lambda |\alpha_{\omega}|^2} \right) = \lambda \sum_{\omega} |O_{\omega}|^2 |\alpha_{\omega}|^2. \tag{188}$$

The limiting generating function

$$\begin{aligned} \lim_{N \rightarrow \infty} \left(\sum_{\omega} |O_{\omega}|^2 e^{\lambda \frac{1}{N} |\alpha_{\omega}|^2} \right)^N &= \lim_{q \rightarrow 1} \exp \left(\frac{1}{1-q} \ln \left(\sum_{\omega} |O_{\omega}|^2 e^{(1-q)\lambda |\alpha_{\omega}|^2} \right) \right) \\ &= \exp \left(\lambda \sum_{\omega} |O_{\omega}|^2 |\alpha_{\omega}|^2 \right) \end{aligned} \quad (189)$$

generates the Poisson distribution

$$p(n, \infty) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \exp \left(\lambda \sum_{\omega} |O_{\omega}|^2 |\alpha_{\omega}|^2 \right) \Big|_{\lambda=-1}. \quad (190)$$

Asymptotically, for large N , the gas of coherent-state indefinite-frequency oscillators possesses Poissonian statistics of excitations. What is interesting the parameter of the Poisson distribution is not just $|\alpha_{\omega}|^2$ but $|O_{\omega}|^2 |\alpha_{\omega}|^2$, a result of great importance for my formulation of quantum field theory. For finite N the distribution is a Rényi-deformed Poissonian.

Let me summarize this section. For finite N we get

$$\langle O, N | \exp \left(\lambda \sum_{\omega} |\alpha_{\omega}|^2 I_{\omega}(N) \right) | O, N \rangle. \quad (191)$$

The $N \rightarrow \infty$ limiting case is

$$\exp \left(\lambda \sum_{\omega} p_{\omega} |\alpha_{\omega}|^2 \right), \quad p_{\omega} = |O_{\omega}|^2. \quad (192)$$

The standard formalism based on infinitely many standard oscillators would imply

$$\exp \left(\lambda \sum_{\omega} |\alpha_{\omega}|^2 \right) \quad (193)$$

which, in practice, is often replaced with

$$\exp \left(\lambda \sum_{\omega} \chi_{\omega} |\alpha_{\omega}|^2 \right), \quad (194)$$

where $0 \leq \chi_{\omega} \leq 1$, $\lim_{\omega \rightarrow \infty} \chi_{\omega} = 0$, is a cut-off function introduced by hand if $\sum_{\omega} |\alpha_{\omega}|^2 = \infty$. So our p_{ω} has *automatically* appeared in the place where standard formalism is artificially amended by adding χ_{ω} .

Remark: The limit $q \rightarrow 1$ is known in information theory as the Shannon limit because the Kolmogorov-Nagumo average of random variable $\ln(1/|O_{\omega}|^2)$ (the amount of information obtained by observation of an event whose probability is $|O_{\omega}|^2$),

$$\begin{aligned} \frac{1}{1-q} \ln \left(\sum_{\omega} |O_{\omega}|^2 e^{(1-q) \ln(1/|O_{\omega}|^2)} \right) &= \frac{1}{1-q} \ln \left(\sum_{\omega} |O_{\omega}|^2 e^{\ln |O_{\omega}|^2 (q-1)} \right) \\ &= \frac{1}{1-q} \ln \left(\sum_{\omega} (|O_{\omega}|^2)^q \right) \end{aligned}$$

tends to Shannon's entropy

$$\lim_{q \rightarrow 1} \frac{1}{1-q} \ln \left(\sum_{\omega} |O_{\omega}|^2 e^{(1-q) \ln(1/|O_{\omega}|^2)} \right) = \sum_{\omega} |O_{\omega}|^2 \ln(1/|O_{\omega}|^2) = - \sum_{\omega} |O_{\omega}|^2 \ln |O_{\omega}|^2.$$

Generalized entropy $\frac{1}{1-q} \ln \left(\sum_{\omega} p_{\omega}^q \right)$ is termed the Rényi q -entropy of probability distribution p_{ω} . Similarly to Shannon's entropy it is additive for independent events

$$\begin{aligned} \frac{1}{1-q} \ln \left(\sum_{\omega_1, \omega_2} (p_{\omega_1} \tilde{p}_{\omega_2})^q \right) &= \frac{1}{1-q} \ln \left(\sum_{\omega_1} p_{\omega_1}^q \sum_{\omega_2} \tilde{p}_{\omega_2}^q \right) \\ &= \frac{1}{1-q} \ln \left(\sum_{\omega_1} p_{\omega_1}^q \right) + \frac{1}{1-q} \ln \left(\sum_{\omega_2} \tilde{p}_{\omega_2}^q \right). \end{aligned}$$

A less obvious property of the Kolmogorov-Nagumo-Rényi average is the following analog of

$$\langle A + C \rangle = \langle A \rangle + C, \quad (195)$$

for a constant C and random variable A :

$$\begin{aligned} \langle A + C \rangle_q &= \frac{1}{1-q} \ln \left(\sum_{\omega} p_{\omega} e^{(1-q)(A_{\omega} + C)} \right) \\ &= \frac{1}{1-q} \ln \left(e^{(1-q)C} \sum_{\omega} p_{\omega} e^{(1-q)A_{\omega}} \right) \\ &= \frac{1}{1-q} \ln \left(\sum_{\omega} p_{\omega} e^{(1-q)A_{\omega}} \right) + \frac{1}{1-q} \ln \left(e^{(1-q)C} \right) \\ &= \langle A \rangle_q + C. \end{aligned} \quad (196)$$

General Kolmogorov-Nagumo averages corresponding to a monotonic function ϕ are defined as

$$\langle A \rangle_{\phi} = \phi^{-1} \left(\sum_{\omega} p_{\omega} \phi(A_{\omega}) \right). \quad (197)$$

It is interesting that only for exponential or linear ϕ one finds $\langle A + C \rangle_{\phi} = \langle A \rangle_{\phi} + C$. \blacktriangle

VII. DIGRESSION ON FREE SCALAR FIELDS

Consider a free scalar field of mass m . The latter means that in momentum (i.e. Fourier) space the field is defined on the manifold of 4-momenta p satisfying the constraint $p^2 = p_0^2 - \mathbf{p}^2 = m^2$ (in sections dealing with relativistic fields I work in units where $c = 1$ and $\hbar = 1$).

Due to the constraint the four components of p are not independent: $p_0 = \pm\sqrt{\mathbf{p}^2 + m^2}$. The manifold is a 3-dimensional hyperboloid in \mathbb{R}^4 , consisting of two sheets corresponding to the two signs of p_0 . Depending on the sign we speak of hyperboloids of, respectively, future-pointing ($p_0 > 0$) and past-pointing ($p_0 < 0$) 4-momenta. If $m = 0$ the hyperboloid is termed the light cone.

Lorentz transformations are linear transformations $p' = Lp$ that do not change $p^2 \neq 0$. If $p^2 = 0$, with $p \neq 0$, the (conformal) group of transformations is larger than the Lorentz group. It contains, in particular, rescalings of the form $p' = 5p$ and the like. If $p = 0$ then all linear transformations preserve p^2 . For any Lorentz transformation $\det L = \pm 1$, a condition implying that $dp'_0 d^3 p' = d^4 p' = \pm d^4 p = \pm dp_0 d^3 p$.

Now assume that $p_0 = \sqrt{\mathbf{p}^2 + m^2} > 0$. Changing variables from (p_0, \mathbf{p}) to (m, \mathbf{p}) we find that

$$d^4 p = dm d^3 p \frac{m}{\sqrt{\mathbf{p}^2 + m^2}} = d(m^2) \frac{d^3 p}{2\sqrt{\mathbf{p}^2 + m^2}}. \quad (198)$$

Since $(p')^2 = (Lp)^2 = p^2 = m^2$ it follows that

$$\begin{aligned} d^4 p &= d(m^2) \frac{d^3 p}{2\sqrt{\mathbf{p}^2 + m^2}} = \pm d^4 p' \\ &= \pm d(m^2) \frac{d^3 p'}{2\sqrt{\mathbf{p}'^2 + m^2}} \end{aligned} \quad (199)$$

and thus (for any m , even $m = 0$)

$$\frac{d^3 p}{2\sqrt{\mathbf{p}^2 + m^2}} = \pm \frac{d^3 p'}{2\sqrt{\mathbf{p}'^2 + m^2}}, \quad (200)$$

if the *four* components of p and p' are related by a Lorentz transformation. The same argument can be applied on the past-pointing part of the mass- m hyperboloid — just change variables from (p_0, \mathbf{p}) to $(-m, \mathbf{p})$.

In consequence, in any integral we can change variables according to the recipe

$$\int_{\mathbb{R}^3} \frac{d^3 p}{2\sqrt{\mathbf{p}^2 + m^2}} F(\mathbf{p}) = \int_{\mathbb{R}^3} \frac{d^3 p'}{2\sqrt{\mathbf{p}'^2 + m^2}} F(\mathbf{p}') = \int_{\mathbb{R}^3} \frac{d^3 p}{2\sqrt{\mathbf{p}^2 + m^2}} F(L\mathbf{p}). \quad (201)$$

The Lorentz invariant measure on mass- m hyperboloid (divided by $(2\pi)^3$ for certain reasons related to Fourier transforms) will be denoted by dp , i.e.

$$dp = \frac{d^3 p}{(2\pi)^3 2\sqrt{\mathbf{p}^2 + m^2}}. \quad (202)$$

In many applications it is convenient to work from the outset with 3D integrals involving dp and not just d^3p . Now let $px = p_0x_0 - \mathbf{p} \cdot \mathbf{x}$ (I will sometimes denote px by $p \cdot x$ or $p_\mu x^\mu$). Two arbitrary solutions

$$\phi_1(x) = \int dp \left(a_1(\mathbf{p}) e^{-ipx} + b_1(\mathbf{p})^\dagger e^{ipx} \right), \quad (203)$$

$$\phi_2(x) = \int dp \left(a_2(\mathbf{p}) e^{-ipx} + b_2(\mathbf{p})^\dagger e^{ipx} \right), \quad (204)$$

of the Klein-Gordon wave equation

$$(\square + m^2)\phi(x) = 0, \quad (205)$$

lead to a family of conserved Noether currents, $\partial^\mu T_\mu{}^r = 0$,

$$\begin{aligned} T_\mu{}^r(x) = & C \left(\partial^r \phi_1(x) \partial_\mu \phi_2(x) + \partial_\mu \phi_1(x) \partial^r \phi_2(x) \right) \\ & - C \left(\partial_\nu \phi_1(x) \partial^\nu \phi_2(x) - m^2 \phi_1(x) \phi_2(x) \right) g_\mu{}^r. \end{aligned} \quad (206)$$

We interpret $a(\mathbf{p})$ and $b(\mathbf{p})$ as amplitudes describing particles and antiparticles, respectively. C is a normalization constant [$C = 1$ for charged fields, $C = 1/2$ for neutral fields i.e. those whose particles equal to their antiparticles: $a(\mathbf{p}) = b(\mathbf{p})$]. The 4-vector

$$P_\mu[\phi_1, \phi_2] = \int d^3x T_{\mu 0}(x_0, \mathbf{x}) \quad (207)$$

$$= \int dp p_\mu \left(b_1(\mathbf{p})^\dagger a_2(\mathbf{p}) + a_1(\mathbf{p}) b_2(\mathbf{p})^\dagger \right) \quad (208)$$

is time-independent. Another conserved current is

$$j_\mu(x) = iq\phi_1(x)\partial_\mu\phi_2(x) - iq\partial_\mu\phi_1(x)\phi_2(x) \quad (209)$$

with the time-independent scalar

$$\hat{q}[\phi_1, \phi_2] = \int d^3x J_0(x_0, \mathbf{x}) \quad (210)$$

$$= q \int dp \left(b_1(\mathbf{p})^\dagger a_2(\mathbf{p}) - a_1(\mathbf{p}) b_2(\mathbf{p})^\dagger \right). \quad (211)$$

The transition from (207) to (208), and from (210) to (211), involves the assumption that it is justified to use the Fourier transform representation of Dirac's delta,

$$\delta^{(3)}(\mathbf{p}) = \frac{1}{(2\pi)^3} \int d^3x e^{i\mathbf{p} \cdot \mathbf{x}}, \quad (212)$$

which I will comment on later. (Of course, one could start with (208), (211), and not with, (207), (210); then the comment on applicability of (212) would be irrelevant.)

Inserting

$$\phi_2(x) = \phi(x) = \int dp \left(a(\mathbf{p}) e^{-ipx} + b(\mathbf{p})^\dagger e^{ipx} \right), \quad (213)$$

$$\phi_1(x) = \phi(x)^\dagger = \int dp \left(b(\mathbf{p}) e^{-ipx} + a(\mathbf{p})^\dagger e^{ipx} \right), \quad (214)$$

we find the 4-momentum

$$P_\mu[\phi^\dagger, \phi] = \int dp p_\mu \left(a(\mathbf{p})^\dagger a(\mathbf{p}) + b(\mathbf{p}) b(\mathbf{p})^\dagger \right) \quad (215)$$

and charge

$$\hat{q}[\phi^\dagger, \phi] = q \int dp \left(a(\mathbf{p})^\dagger a(\mathbf{p}) - b(\mathbf{p}) b(\mathbf{p})^\dagger \right). \quad (216)$$

What one usually finds in the literature are the particular cases (213) and (214), while the general ones are not mentioned. The general expressions are complex (or non-Hermitian, if quantized) and this is probably why they may seem “unphysical”, although their real and imaginary (Hermitian and anti-Hermitian, respectively) parts are separately conserved.

However, when one comes to field quantization it may pay to maintain their general forms. Let me give several applications.

A. Bosonic Fock space

Let us begin with the orthodox formulation based on bosonic Fock space representation: $a(\mathbf{p}) = \sum_j a_j \psi_j(\mathbf{p})$, $b(\mathbf{p}) = \sum_j b_j \psi_j(\mathbf{p})$. a_j and b_j are bosonic annihilation operators and $\psi_j(\mathbf{p})$ are orthogonal functions

$$\int dp \overline{\psi_j(\mathbf{p})} \psi_k(\mathbf{p}) = \delta_{jk}. \quad (217)$$

Then

$$P_\mu[\phi^\dagger, \phi] = \int d^3x T_{\mu 0}(x_0, \mathbf{x}) \quad (218)$$

$$= \int dp p_\mu \left(a(\mathbf{p})^\dagger a(\mathbf{p}) + b(\mathbf{p}) b(\mathbf{p})^\dagger \right) \quad (219)$$

$$= \sum_{jk} \int dp p_\mu \left(\overline{\psi_j(\mathbf{p})} \psi_k(\mathbf{p}) a_j^\dagger a_k + \psi_k(\mathbf{p}) \overline{\psi_j(\mathbf{p})} b_k b_j^\dagger \right) \quad (220)$$

$$= \sum_{jk} \int dp p_\mu \overline{\psi_j(\mathbf{p})} \psi_k(\mathbf{p}) \frac{1}{2} \left(a_j^\dagger a_k + b_k b_j^\dagger \right) \quad (221)$$

$$= \sum_{jk} (P_\mu)_{jk} \left(a_j^\dagger a_k + b_k b_j^\dagger \right) \quad (222)$$

$$= \sum_{jk} \left(a_j^\dagger (P_\mu)_{jk} a_k + b_j^\dagger (P_\mu)_{jk} b_k \right) + \sum_j (P_\mu)_{jj}. \quad (223)$$

$$\hat{q}[\phi^\dagger, \phi] = \int d^3x J_0(x_0, \mathbf{x}) \quad (224)$$

$$= q \int dp \left(a(\mathbf{p})^\dagger a(\mathbf{p}) - b(\mathbf{p}) b(\mathbf{p})^\dagger \right) \quad (225)$$

$$= q \sum_j \left(a_j^\dagger a_j - b_j^\dagger b_j \right) - q \sum_j 1. \quad (226)$$

The “vacuum terms” $\sum_j (P_\mu)_{jj}$ and $-q \sum_j 1$ are divergent. The infinite negative charge of vacuum may be intuitively interpreted as the charge of the “Dirac sea” of antiparticles. The remaining charge term

$$: \hat{q} : = \hat{q} - \hat{q}_{\text{vacuum}} = q \sum_j \left(a_j^\dagger a_j - b_j^\dagger b_j \right) = q(n_+ - n_-) \quad (227)$$

looks reasonable.

A similar analysis can be performed for the case of 4-momentum. For example, the Hamiltonian $H = P_0$ leads to vacuum energy

$$H_{\text{vacuum}} = \sum_j \int dp \sqrt{\mathbf{p}^2 + m^2} |\psi_j(\mathbf{p})|^2 > m \sum_j \int dp |\psi_j(\mathbf{p})|^2 = m \sum_j 1. \quad (228)$$

On the other hand

$$: H : = H - H_{\text{vacuum}} = \sum_{jk} \left(a_j^\dagger H_{jk} a_k + b_j^\dagger H_{jk} b_k \right) \quad (229)$$

has the form we know from general considerations on the Fock space \mathcal{F} . In spite of these partly acceptable predictions the presence of infinite terms makes both 4-momentum and

charge ill defined. “Sensible mathematics” does not allow us to ignore these infinities “because we do not want them”. And even if we ignore the infinities at this stage, they will be reappearing again and again in various (practically all) calculations. Quantization based on the Fock space simply does not correctly work, although in some sense it must be close to a correct theory.

B. Reducible representation of HOLA

Now assume that

$$a_1(\mathbf{p}) = |\mathbf{p}\rangle\langle\mathbf{p}| \otimes \hat{b}(\mathbf{p}), \quad (230)$$

$$b_1(\mathbf{p}) = |\mathbf{p}\rangle\langle\mathbf{p}| \otimes \hat{a}(\mathbf{p}), \quad (231)$$

$$a_2(\mathbf{p}) = 1 \otimes \hat{a}(\mathbf{p}), \quad (232)$$

$$b_2(\mathbf{p}) = 1 \otimes \hat{b}(\mathbf{p}). \quad (233)$$

I do not specify at this moment the exact mathematical meanings of $|\mathbf{p}\rangle\langle\mathbf{p}|$ and the hatted operators — we will return to it. We obtain

$$\begin{aligned} P_\mu[\phi_1, \phi_2] &= \int dp p_\mu \left(b_1(\mathbf{p})^\dagger a_2(\mathbf{p}) + a_1(\mathbf{p}) b_2(\mathbf{p})^\dagger \right) \\ &= \int dp p_\mu |\mathbf{p}\rangle\langle\mathbf{p}| \otimes \left(\hat{a}(\mathbf{p})^\dagger \hat{a}(\mathbf{p}) + \hat{b}(\mathbf{p}) \hat{b}(\mathbf{p})^\dagger \right) \\ &= \int dp p_\mu |\mathbf{p}\rangle\langle\mathbf{p}| \otimes \left(\hat{a}(\mathbf{p})^\dagger \hat{a}(\mathbf{p}) + \hat{b}(\mathbf{p})^\dagger \hat{b}(\mathbf{p}) \right) + \int dp p_\mu |\mathbf{p}\rangle\langle\mathbf{p}| \otimes \iota(\mathbf{p}, \mathbf{p}), \end{aligned} \quad (234)$$

$$\begin{aligned} \hat{q}[\phi_1, \phi_2] &= q \int dp \left(b_1(\mathbf{p})^\dagger a_2(\mathbf{p}) - a_1(\mathbf{p}) b_2(\mathbf{p})^\dagger \right) \\ &= q \int dp |\mathbf{p}\rangle\langle\mathbf{p}| \otimes \left(\hat{a}(\mathbf{p})^\dagger \hat{a}(\mathbf{p}) - \hat{b}(\mathbf{p}) \hat{b}(\mathbf{p})^\dagger \right) \\ &= q \int dp |\mathbf{p}\rangle\langle\mathbf{p}| \otimes \left(\hat{a}(\mathbf{p})^\dagger \hat{a}(\mathbf{p}) - \hat{b}(\mathbf{p})^\dagger \hat{b}(\mathbf{p}) \right) - q \int dp |\mathbf{p}\rangle\langle\mathbf{p}| \otimes \iota(\mathbf{p}, \mathbf{p}), \end{aligned} \quad (235)$$

where $\iota(\mathbf{p}, \mathbf{p}) = [\hat{b}(\mathbf{p}), \hat{b}(\mathbf{p})^\dagger]$ is the analogue of (129). The resulting operators are, of course, Hermitian.

As we can see, the 4-momentum and charge involve number operators of the form (124). Several approaches to relativistic field theoretic formalism, based on $\iota(\mathbf{p}, \mathbf{p}) = 1$, were described in detail in [24–27]. Later on in these notes we will reconsider all the steps involved in such a choice of $\iota(\mathbf{p}, \mathbf{p})$.

C. Sequential approach

The next application is the following. Assume we have two sequences of operators, $a(\mathbf{p}, n)$, $b(\mathbf{p}, n)$, convergent in some sense to $a(\mathbf{p})$ and $b(\mathbf{p})$. Now take

$$\begin{aligned}\phi_2(x) &= \int dp \left(a(\mathbf{p}, n_2) e^{-ipx} + b(\mathbf{p}, n_2)^\dagger e^{ipx} \right), \\ \phi_1(x) &= \int dp \left(b(\mathbf{p}, n_1) e^{-ipx} + a(\mathbf{p}, n_1)^\dagger e^{ipx} \right).\end{aligned}$$

Then

$$P_\mu[\phi_1, \phi_2] = \int d^3x T_{\mu 0}(x_0, \mathbf{x}; n_1, n_2) \quad (236)$$

$$= \int dp p_\mu \left(a(\mathbf{p}, n_1)^\dagger a(\mathbf{p}, n_2) + b(\mathbf{p}, n_1) b(\mathbf{p}, n_2)^\dagger \right), \quad (237)$$

$$\hat{q}[\phi_1, \phi_2] = \int d^3x J_0(x_0, \mathbf{x}; n_1, n_2) \quad (238)$$

$$= q \int dp \left(a(\mathbf{p}, n_1)^\dagger a(\mathbf{p}, n_2) - b(\mathbf{p}, n_1) b(\mathbf{p}, n_2)^\dagger \right). \quad (239)$$

are time independent for all n_1 and n_2 . We will later consider situations where the Fourier form of Dirac delta (212) is rigorously applicable in (236) and (238) if n_1 and n_2 are finite. Integration in (236) and (238) must be then performed before the limits $n_1 \rightarrow \infty$, $n_2 \rightarrow \infty$ are evaluated. Alternatively, the limits $n_1 \rightarrow \infty$, $n_2 \rightarrow \infty$ may be performed under momentum integrals in (237) and (239), but not under position integrals in (236) and (238).

VIII. SEQUENTIAL APPROACH TO DIRAC DELTAS

Dirac described his $\delta(x)$ as follows: “To get a picture of $\delta(x)$, take a function of the real variable x which vanishes everywhere except inside a small domain, of length ε say, surrounding the origin $x = 0$, and which is so large inside this domain that its integral over this domain is unity. The exact shape of the function inside this domain does not matter, provided there are no unnecessarily wild variations (for example provided the function is always of order ε^{-1}). Then in the limit $\varepsilon \rightarrow 0$ this function will go over into $\delta(x)$ ” [28].

One possible meaning of Dirac’s delta was formalized by Laurent Schwartz [29], who interpreted it as an “evaluation-at-zero map”, i.e. a linear functional that, given a function f , returns its value at $x = 0$. In the standard functional-analysis notation we would write

$\langle \delta | f \rangle = f(0)$ or, more generally, $\langle \delta_x | f \rangle = f(x)$; in the bra-ket notation of Dirac one would express the same as $\langle x | f \rangle = f(x)$.

An alternative to the Schwartz formalism was developed by the Polish mathematician Jan Mikusiński [30] and his coworkers, a program that culminated in the textbook [31]. The authors wanted to make precise the intuition of Dirac that

$$\int_{-\infty}^{\infty} dx \delta(x) f(x) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx \delta_n(x) f(x), \quad (240)$$

and, on this basis, formulate the whole of theory of distributions. In introduction to [31] they wrote: “We shall not avail ourselves to the methods of functional analysis and we shall not define distributions as functionals. In applied mathematics distributions are regarded as ordinary functions, e.g. the function $\delta(x)$ of Dirac. Essentially, however, distributions are not functions but in an intuitive sense, they may be approximated by functions. Approximation, strictly defined, is our starting point for the definition of distributions.”

The idea of “filtering integrals”, satisfying $\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx \delta_n(x) f(x) = [f(0_-) + f(0_+)]/2$, had been known to Cauchy, Hermite, Poisson, Kirchhoff, and Heaviside many decades before Dirac rediscovered the notion for the purposes of quantum mechanics (for a review see the fifth chapter of [32]). The Mikusiński sequential approach goes deeper, generalizing the idea to general distributions. Distributions are in this formalism equivalence classes of fundamental sequences of ordinary functions, much the same way as in Cantor’s theory real numbers are defined as equivalence classes of fundamental sequences of rational numbers. It follows that one first has to define fundamental sequences of functions, and then introduce an equivalence relation that leads to equivalence classes.

Let me begin with recalling some basic notions. We say that f_n converges uniformly to f , if for any $\varepsilon > 0$ one can find n_0 such that $|f_n(x) - f(x)| < \varepsilon$ for any $n > n_0$. The sequence $f_n(x) = x/n$ is not uniformly convergent to $f(x) = 0$, since no matter what $\varepsilon > 0$ and $n < \infty$ we take, we will always find x for which $|x/n - 0| > \varepsilon$ (just take any $x > n\varepsilon$). We say that f_n converges in the interval $A < x < B$ almost uniformly to f , if it converges to f uniformly on each finite closed interval contained in the interval $A < x < B$ (this definition of almost uniform convergence is employed in [31]). The sequence $f_n(x) = x/n$ is almost uniformly convergent to $f(x) = 0$ on the whole real axis since x/n converges uniformly to 0 on each interval $-\infty < a \leq x \leq b < \infty$ (now $|x|$ cannot be arbitrarily large and $|x/n - 0| < \varepsilon$, for each x in the interval, if n is sufficiently large).

In part I of [31] a delta-sequence is defined as any sequence δ_n of continuous functions, satisfying

$$\int_{-\infty}^{\infty} dx \delta_n(x) = 1, \quad (241)$$

$$\delta_n(x) = F_n''(x), \quad (242)$$

where F_n are twice differentiable functions almost uniformly convergent to

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}, \quad (243)$$

and the prime denotes the derivative.

The intuition behind the construction is the following. First,

$$F'(x) = \theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}. \quad (244)$$

The derivatives at 0 are defined by right and left limits. $\theta'(x)$ is zero everywhere, with the exception of $x = 0$, where $\theta(x)$ “jumps from 0 to 1 infinitely fast” — this is, roughly, how Dirac imagined his delta “function”.

A nontrivial and useful example of a delta-sequence is obtained by means of Cauchy’s principal value,

$$\int_{-\infty}^{\infty} dx f(x) \frac{e^{ixn}}{x} = \int_{-\infty}^{\infty} dx f(x/n) \frac{e^{ix}}{x} = \lim_{\epsilon \rightarrow 0+} \left(\int_{-1/\epsilon}^{-\epsilon} + \int_{\epsilon}^{1/\epsilon} \right) dx f(x/n) \frac{e^{ix}}{x}. \quad (245)$$

If $f(x)$ does not grow too fast with $x \rightarrow \pm\infty$ (i.e. when for large n the function $f(x/n)$ is sufficiently slowly changing if compared to $1/x$) then

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx f(x/n) \frac{e^{ix}}{x} &= f(0) \lim_{\epsilon \rightarrow 0+} \left(\int_{-1/\epsilon}^{-\epsilon} + \int_{\epsilon}^{1/\epsilon} \right) dx \frac{e^{ix}}{x} \\ &= i\pi f(0) = i\pi \int_{-\infty}^{\infty} dx f(x) \delta(x). \end{aligned} \quad (246)$$

In this sense,

$$\frac{e^{ixn}}{i\pi x} = \delta_n(x). \quad (247)$$

As we can see, delta-sequences can be given also by complex-valued functions.

In parts II and III of the book the authors further restrict delta-sequences to the narrower class of those δ_n that are smooth, and $\delta_n(x) = 0$ for $|x| > \alpha_n$, where α_n is a sequence of

real numbers convergent to 0 (now $\delta_n(x)$ is exactly vanishing outside a given interval — the shorter, the greater n). The authors explain in the introduction that there are certain differences between different parts of the book since they correspond to different versions of the theory, developed by the Mikusiński group over different time periods. In particular, the requirement of smoothness and exact vanishing of $\delta_n(x)$ outside of an interval is *not needed* for (240) to hold, but turns out convenient for some applications.

I stress the latter property of the Mikusiński-Antosik-Sikorski approach since in what follows I want to perform a similar step and restrict admissible delta-sequences to functions with some specified properties.

Now, as we have developed some intuitions for the sequential approach, let us describe in more detail the equivalence relation that defines Dirac's delta. We say that the sequence of continuous functions f_n defined for $A < x < B$ is fundamental if there exists an integer $k \geq 0$ and an almost uniformly convergent sequence F_n of functions, satisfying $d^k F_n(x)/dx^k = f_n(x)$. $k = 2$ for δ_n . The sequences f_n and g_n are equivalent (we write $f_n \sim g_n$), if $d^k F_n(x)/dx^k = f_n(x)$, $d^k G_n(x)/dx^k = g_n(x)$, for the same k , and both F_n and G_n converge almost uniformly to the same function.

One can prove (cf. p. 11 in [31]) that \sim is an equivalence relation. The equivalence class $[f_n] =: f$ is the distribution associated with equivalent fundamental sequences. Any f_n from the equivalence class is a representative of the distribution f .

Let us explicitly show an appropriate construction of δ . Let δ_n be the function of the type shown in Fig. 1. It is clear that plots similar to the middle and lower ones would be obtained if one replaced this concrete form of δ_n by any function with the same support (and integrable to unity), or by a sufficiently narrow Gaussian supported on the whole of \mathbb{R} . All such delta-sequences thus belong to the same equivalence class $\delta = [\delta_n]$. Delta sequences that tend to infinity at $x = 0$ will be termed Λ -shaped.

IX. FURTHER SPLITTING OF EQUIVALENCE CLASSES: DISCONTINUITIES AND DELTA-SEQUENCES REGULAR AT ZERO

In Eq. (240) I have purposefully avoided one more natural identification, namely

$$\int_{-\infty}^{\infty} dx \delta(x) f(x) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx \delta_n(x) f(x) = f(0) = \langle \delta | f \rangle. \quad (248)$$

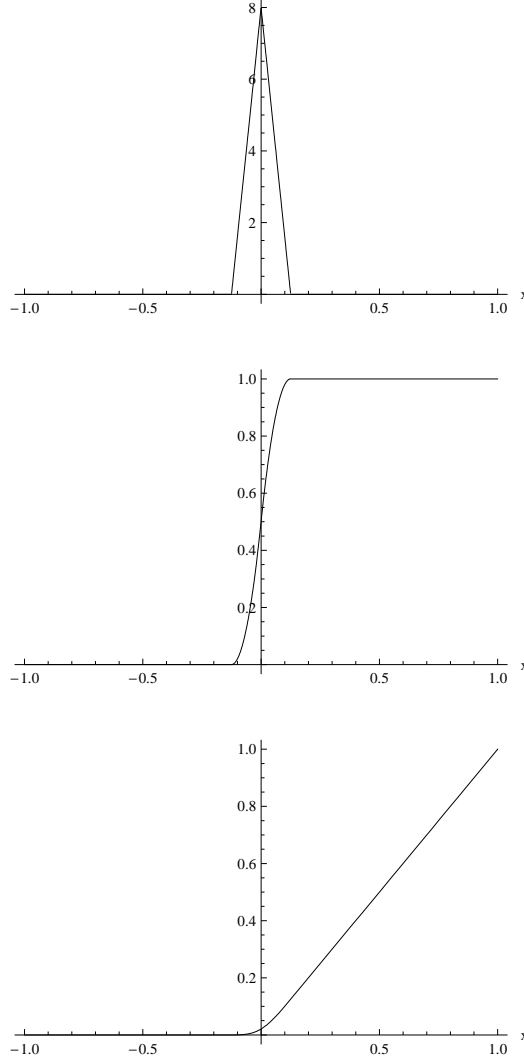


FIG. 1: Λ -shaped $\delta_n(x)$ (upper), $\theta_n(x) = \int_{-\infty}^x dx_1 \delta_n(x_1)$ (middle), and $F_n(x) = \int_{-\infty}^x dx_2 \int_{-\infty}^{x_2} dx_1 \delta_n(x_1)$ (lower). $\delta_n(x) = F_n''(x)$. The support of $\delta_n(x)$ is here given by the interval $[-\frac{1}{n}, \frac{1}{n}]$ with $n = 8$. F_n uniformly converges to F given by (243).

The reason for this omittance is that (248) is true only for functions continuous at $x = 0$. In case of discontinuity the delta-sequence shown at Fig. 1 would imply

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx \delta_n(x) f(x) = \frac{f(0_-) + f(0_+)}{2}, \quad (249)$$

where $f(0_{\pm})$ denotes the left and right limits of $f(x)$ at $x = 0$.

But what would have happened had we replaced this concrete delta-sequence by a new (fundamental) sequence $\tilde{\delta}_n(x) = \delta_n(x - \frac{1}{n})$? All the plots from Fig. 1 would be simply shifted

to the right by $1/n$, so this is again a delta-sequence, but

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx \tilde{\delta}_n(x) f(x) = f(0_+). \quad (250)$$

Of particular interest are also the following two examples (M-shaped delta sequence, Fig. 2)

$$\tilde{\delta}_n(x) = \frac{1}{2} \delta_n\left(x - \frac{1}{n}\right) + \frac{1}{2} \delta_n\left(x + \frac{1}{n}\right) \quad (251)$$

and ($\Lambda\Lambda$ -shaped delta sequence, Fig. 3)

$$\tilde{\delta}_n(x) = \frac{1}{2} \delta_n\left(x - \frac{2}{n}\right) + \frac{1}{2} \delta_n\left(x + \frac{2}{n}\right), \quad (252)$$

both yielding

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx \tilde{\delta}_n(x) f(x) = \frac{f(0_-) + f(0_+)}{2}. \quad (253)$$

It is clear that the equivalence relation we have discussed in the previous section can only be true if Dirac deltas “act” on continuous f . Schwartz and the other authors go typically even further and assume that one deals with functions that are smooth.

However, in quantum field theory one encounters expressions of the form

$$\int_{-\infty}^{\infty} dx \delta(x) \delta(x) f(x) \quad (254)$$

(or even worse), involving “functions” $\delta(x)f(x)$ that are quite far from any form of continuity. This is one of the sources of the infinities that plague “the so-called quantum electrodynamics” (the phrase of Dirac [1]), especially in loop-amplitude calculations. Intuitively one expects that (254) equals $\delta(0)f(0)$, whatever this means. It is interesting that such a rule can be indeed derived from dimensional regularization techniques for path integrals, but $\delta(0)$ is then treated as the “infinite quantity $\int dk/(2\pi)$ ” [33].

From a sequential point of view an appropriate calculation could read

$$\int_{-\infty}^{\infty} dx \delta(x) \delta(x) f(x) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} dx \delta_n(x) \tilde{\delta}_m(x) f(x) \quad (255)$$

$$= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx \delta_n(x) \tilde{\delta}_m(x) f(x) \quad (256)$$

$$= \lim_{m \rightarrow \infty} \tilde{\delta}_m(0) f(0) \quad (257)$$

$$= \lim_{n \rightarrow \infty} \delta_n(0) f(0), \quad (258)$$

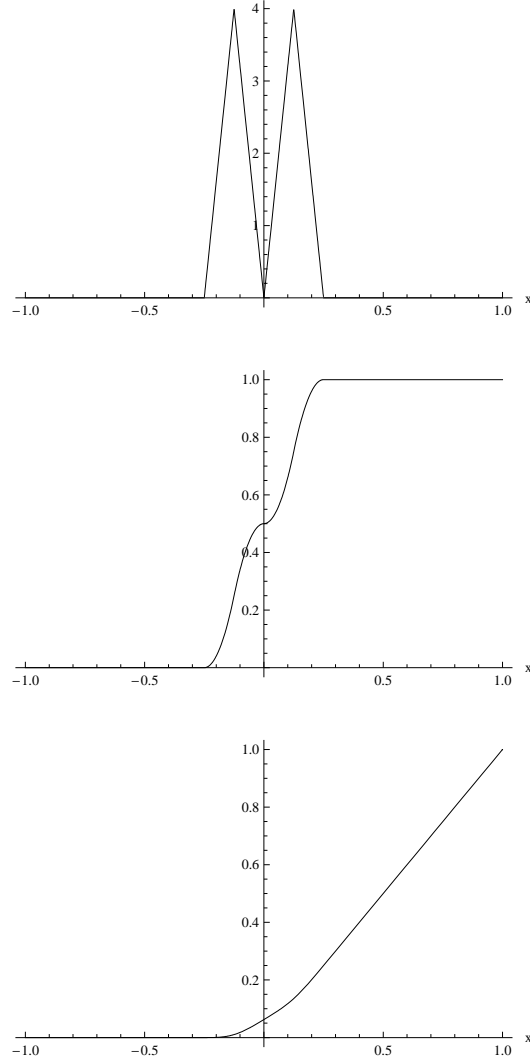


FIG. 2: Analogues of the plots from Fig. 1 but for M-shaped delta-sequence (251). $F_n''(0) = 0$ but the third derivative is not continuous at $x = 0$. F_n uniformly converges to F given by (243).

where $\delta_n(x)$ and $\tilde{\delta}_n(x)$ are, in principle, different representatives of $\delta = [\delta_n] = [\tilde{\delta}_n]$. It is obvious that in order that the expression be well defined one has to require

$$\lim_{n \rightarrow \infty} \tilde{\delta}_n(0)f(0) = \lim_{n \rightarrow \infty} \delta_n(0)f(0). \quad (259)$$

The examples of delta-sequences we have discussed above would imply

$$\lim_{n \rightarrow \infty} \delta_n(0)f(0) = \infty \times \frac{f(0_-) + f(0_+)}{2}, \quad (260)$$

$$\lim_{n \rightarrow \infty} \tilde{\delta}_n(0)f(0_+) = 0 \times f(0_+), \quad (261)$$

$$\lim_{n \rightarrow \infty} \tilde{\delta}_n(0) \frac{f(0_-) + f(0_+)}{2} = 0 \times \frac{f(0_-) + f(0_+)}{2}, \quad (262)$$

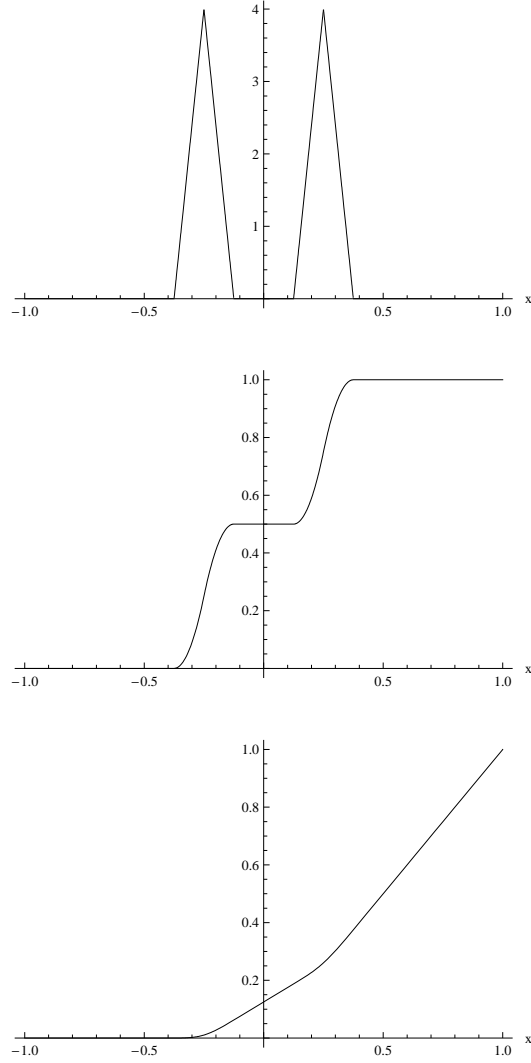


FIG. 3: Analogues of the plots from Fig. 1 but for $\Lambda\Lambda$ -shaped delta-sequence (252). F_n uniformly converges to F given by (243), and all derivatives of F'_n vanish at $x = 0$.

so that

$$\int_{-\infty}^{\infty} dx \delta(x) \delta(x) f(x) \quad (263)$$

becomes ambiguous, unless one restricts delta sequences to a subset of $[\delta_n]$. In other words, we have to modify the equivalence relation.

Remark: The proof that δ^2 does not exist, given in [31], is based on the assumption that the sequential definition of (254) should read

$$\int_{-\infty}^{\infty} dx \delta(x) \delta(x) f(x) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx \delta_n(x)^2 f(x), \quad (264)$$

which indeed does not exist for any delta-sequence. It seems that the reason for such a choice of definition lies in ambiguities implied by the form of \sim . However, we are free to change the equivalence relation.▲

One option is the following: Two delta-sequences δ_n and $\tilde{\delta}_n$ are equivalent if $\delta_n \sim \tilde{\delta}_n$ with respect to the Mikusiński-Antosik-Sikorski relation discussed above and, in addition,

$$\lim_{n \rightarrow \infty} \delta_n(0) = \lim_{n \rightarrow \infty} \tilde{\delta}_n(0) =: \delta(0), \quad (265)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx \tilde{\delta}_n(x) f(x) &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dx \delta_n(x) f(x) \\ &= \frac{1}{2} f(0_-) + \frac{1}{2} f(0_+), \end{aligned} \quad (266)$$

for any f . Equivalence classes with respect to this new equivalence relation define different Dirac deltas. Deltas belonging to the same equivalence class can be uniquely multiplied,

$$\int_{-\infty}^{\infty} dx \delta(x)^j f(x) = \lim_{n_1 \rightarrow \infty} \dots \lim_{n_j \rightarrow \infty} \int_{-\infty}^{\infty} dx \delta_{n_1}^{(1)}(x) \dots \delta_{n_j}^{(j)}(x) f(x) \quad (267)$$

$$= \frac{1}{2} \delta(0_-)^{j-1} f(0_-) + \frac{1}{2} \delta(0_+)^{j-1} f(0_+) \quad (268)$$

$$= \delta(0)^{j-1} \frac{1}{2} (f(0_-) + f(0_+)). \quad (269)$$

Here $\delta_{n_1}^{(1)}(x) \dots \delta_{n_j}^{(j)}(x)$ means that we are free to take any (continuous) representatives $\delta_{n_i}^{(i)}(x)$ for any of the delta-sequences. The examples show that there is no difficulty with assuming even $\delta(0) = 0$, an option suggested by relativistic invariance, as we shall see in a moment.

Delta-sequences vanishing at 0 are not a new concept (cf. Chapter V, Eq. (38) in [32]). An intriguing example of such a delta-sequence was recently found in the context of time-of-arrival operator [34, 35]. In spite of this, many authors who generalize the concept of Dirac's delta stick to the “obvious” requirement of either divergence or indefiniteness of $\delta(0)$ (cf. [36–38]).

X. M-SHAPED AND M-SHAPED DIRAC DELTAS

In what follows we will mostly work in “momentum space” so that delta-sequences will depend on arguments p , k , etc., whereas x will be reserved for their Fourier-transform arguments. This is perhaps different from what one is accustomed to, but is more convenient from the point of view of applications we have in mind.

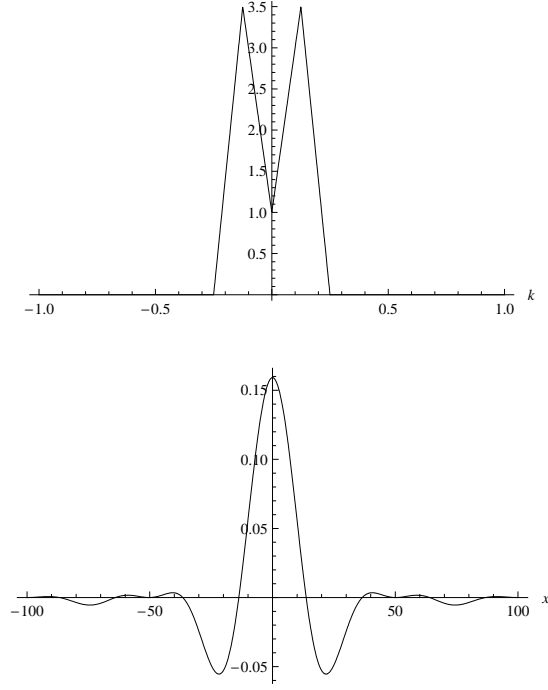


FIG. 4: The M-shaped function (270) with $a = 1$, $\epsilon = 1/2$ (upper), and its Fourier transform (lower).

A. M-shaped delta-sequences

Let us consider the function shown in the upper part of Fig. 4. It is a particular example, for $a = 1$ and $\epsilon = 1/2$, of

$$\delta(k, a, \epsilon) = \begin{cases} 0 & \text{for } k < -\frac{\epsilon}{2} \\ \left(\frac{4k}{\epsilon} + 2\right)\left(\frac{2}{\epsilon} - \frac{a}{2}\right) & \text{for } -\frac{\epsilon}{2} \leq k < -\frac{\epsilon}{4} \\ -\frac{4k}{\epsilon}\left(\frac{2}{\epsilon} - \frac{3a}{2}\right) + a & \text{for } -\frac{\epsilon}{4} \leq k < 0 \\ \frac{4k}{\epsilon}\left(\frac{2}{\epsilon} - \frac{3a}{2}\right) + a & \text{for } 0 \leq k < \frac{\epsilon}{4} \\ \left(-\frac{4k}{\epsilon} + 2\right)\left(\frac{2}{\epsilon} - \frac{a}{2}\right) & \text{for } \frac{\epsilon}{4} \leq k < \frac{\epsilon}{2} \\ 0 & \text{for } \frac{\epsilon}{2} \leq k, \end{cases} \quad (270)$$

($a \geq 0$, $\epsilon > 0$). Its Fourier transform,

$$\begin{aligned}\hat{\delta}(x, a, \epsilon) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(k, a, \epsilon) e^{ikx} dk \\ &= \frac{8\epsilon a + (4 - \epsilon a) \cos \frac{\epsilon x}{4}}{\pi \epsilon^2 x^2} \sin^2 \frac{\epsilon x}{8},\end{aligned}\tag{271}$$

$$\lim_{\epsilon \rightarrow 0} \hat{\delta}(x, a, \epsilon) = \frac{1}{2\pi},\tag{272}$$

is the real function shown in the lower part of Fig. 4 . The sequence $\delta_n(k, a) = \delta(k, a, \frac{1}{n})$, with natural n (i.e. $\epsilon = 1/n$), is an example of what I call an M-shaped delta-sequence, which is again a particular example of the delta-sequence in the sense of [31]. Indeed,

$$\int_{-\infty}^{\infty} \delta(k, a, \frac{1}{n}) dk = 1,\tag{273}$$

and

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(k) \delta(k, a, \frac{1}{n}) dk = \frac{f(0_-) + f(0_+)}{2}.\tag{274}$$

M-shaped delta-sequences do not have to vanish at 0,

$$\delta(0, a, \frac{1}{n}) = a,\tag{275}$$

for all n , so that

$$\lim_{n \rightarrow \infty} \delta(0, a, \frac{1}{n}) = a.\tag{276}$$

For each a we deal with a sequence belonging to a different equivalence class. We will now show that $a = 0$ we have encountered before is an important special case.

B. M-shaped deltas with respect to more general measures

Let us assume that instead of dp we have to use a measure $d\mu(p) = \rho(p)dp$, and an appropriate delta is needed,

$$\int d\mu(p') \delta(p, p') f(p') = f(p),\tag{277}$$

with $\delta(p, p) = a$, say, where a is a constant (that is, not a function of p). The standard solution

$$\delta(p, p') = \rho(p')^{-1} \delta(p - p'),\tag{278}$$

if generalized to M-shaped deltas implies $\delta(p, p) = \rho(p)^{-1}\delta(0)$ and will not lead to a independent of p (unless $\delta(0) = a = 0$).

So let us try a different option. Let $\delta(p, a, \frac{1}{n})$, $\delta(0, a, \frac{1}{n}) = a$, be an arbitrary M-shaped delta-sequence. The sequence

$$\delta_n(p, p') = \rho(p')^{-1}\delta(p - p', a\rho(p), \frac{1}{n}), \quad (279)$$

$\rho(p) = d\mu(p)/dp$, has the required properties

$$\lim_{n \rightarrow \infty} \int d\mu(p') \delta_n(p, p') f(p') = \frac{f(p_-) + f(p_+)}{2}, \quad (280)$$

$$\delta_n(p, p) = a \quad (281)$$

As an exercise, compute in two different ways

$$\begin{aligned} & \int d\mu(p') \delta(p, p') \delta(p + k, p' + k) f(p') \\ &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int dp' \rho(p') \rho(p')^{-1} \delta(p - p', a\rho(p), \frac{1}{n}) \rho(p' + k)^{-1} \delta(p - p', a\rho(p + k), \frac{1}{m}) f(p') \\ &= \lim_{n \rightarrow \infty} \delta(p - p, a\rho(p), \frac{1}{n}) \rho(p + k)^{-1} f(p) \\ &= a\rho(p) \rho(p + k)^{-1} f(p). \end{aligned} \quad (282)$$

Reversing the order of limits we would get

$$\begin{aligned} & \int d\mu(p') \delta(p, p') \delta(p + k, p' + k) f(p') \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int dp' \rho(p') \rho(p')^{-1} \delta(p - p', a\rho(p), \frac{1}{n}) \rho(p' + k)^{-1} \delta(p - p', a\rho(p + k), \frac{1}{m}) f(p') \\ &= \lim_{m \rightarrow \infty} \rho(p + k)^{-1} \delta(p - p, a\rho(p + k), \frac{1}{m}) f(p) \\ &= \rho(p + k)^{-1} a\rho(p + k) f(p) \\ &= a f(p). \end{aligned} \quad (283)$$

Consistency of the two calculations requires that

$$a\rho(p) \rho(p + k)^{-1} = a \quad (284)$$

for all p and k , so that either $a \neq 0$ and then $\rho(p) = \text{const}$, or $\rho(p) \neq \text{const}$ and $a = 0$.

In non relativistic quantum mechanics $\rho(p) = \text{const}$ so that a can be arbitrary. However, relativistic measures involve a nontrivial ρ and thus we have to assume $a = 0$. From now on, if I write of M-shaped delta-sequences, as opposed to the M-shaped ones, I mean those corresponding to $a = 0$. A particular class of M-shaped delta sequences are the $\Lambda\Lambda$ -shaped sequences whose derivatives of all orders vanish at 0.

C. M-shaped versus Λ -shaped delta-sequences

Let us discuss in more detail the properties of delta-sequences of the types shown in Figs. 1–3. All of them are derived from a single $\delta_n(k)$, and can be regarded as particular cases of the formula

$$\delta_n(k, j) = \frac{1}{2}\delta_n\left(k - \frac{j}{n}\right) + \frac{1}{2}\delta_n\left(-k - \frac{j}{n}\right) = \delta_n(-k, j), \quad j = 0, 1, 2, \dots \quad (285)$$

The Fourier transform

$$\hat{\delta}_n(x, j) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \delta_n(k, j) e^{ikx} \quad (286)$$

$$= \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \delta_n\left(k - \frac{j}{n}\right) e^{ikx} + \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \delta_n\left(-k - \frac{j}{n}\right) e^{ikx} \quad (287)$$

$$= \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \delta_n(k) e^{ikx} e^{ijx/n} + \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \delta_n(k) e^{-ikx} e^{-ijx/n} \quad (288)$$

$$= \frac{1}{2} \hat{\delta}_n(x) e^{ijx/n} + \frac{1}{2} \overline{\hat{\delta}_n(x)} e^{-ijx/n} \quad (289)$$

is real.

It is instructive to pause here for a moment and perform all these calculations for the explicit choice of Λ -shaped delta-sequence from Fig. 1 (now the argument is k and not x),

$$\delta_n(k) = \begin{cases} 0 & \text{for } k < -\frac{1}{n}, \\ n^2 k + n & \text{for } -\frac{1}{n} \leq k < 0, \\ -n^2 k + n & \text{for } 0 \leq k < \frac{1}{n}, \\ 0 & \text{for } k \geq \frac{1}{n}. \end{cases} \quad (290)$$

Its Fourier transform

$$\begin{aligned} \hat{\delta}_n(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \delta_n(k) e^{ikx} \\ &= \frac{1}{2\pi} \left(\frac{x}{2n}\right)^{-2} \sin^2 \frac{x}{2n} \end{aligned} \quad (291)$$

is real, symmetric, bounded, $|\hat{\delta}_n(x)| \leq 1/(2\pi)$, and almost uniformly (but not uniformly) convergent to $1/(2\pi)$.

For all delta-sequences generated from a symmetric $\delta_n(k) = \delta_n(k, 0)$, $\overline{\hat{\delta}_n(x)} = \hat{\delta}_n(x)$, we can further simplify

$$\hat{\delta}_n(x, j) = \hat{\delta}_n(x) \cos \frac{jx}{n}, \quad (292)$$

$$\lim_{n \rightarrow \infty} \hat{\delta}_n(x, j) = \frac{1}{2\pi} \lim_{n \rightarrow \infty} \cos \frac{jx}{n} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dk \delta_n(k) e^{ikx} = \frac{1}{2\pi}. \quad (293)$$

The latter formulas indicate very clearly that there cannot be much difference between Λ -shaped delta sequences we are accustomed to, and the regular-at-zero M-shaped and $\Lambda\Lambda$ -shaped ones. In fact, they all define the same distribution in the sense of Mikusiński, but $\delta_n(k, 0)$ and $\delta_n(k, j)$, $j > 0$, belong to different equivalence classes with respect to our modified relation. In particular, products of Dirac deltas $\delta(k, j) = [\delta_n(k, j)]$ can be taken for all integer $j > 0$, but $\delta(k, 0)\delta(k, j)$ is ill defined.

If both $f(0_-)$ and $f(0_+)$ are finite, we get $\int dk \delta(k, j)^N f(k) = 0$ for all $N = 2, 3, \dots$ and $j = 1, 2, \dots$. If any of $f(0_\pm)$ is infinite, the expression is still not well defined. This is why a modification of Dirac's delta cannot solve all the problems of field quantization. However, the situation is much better if we additionally take a nontrivially chosen $I(\mathbf{p})$ occurring in HOLA. I will discuss the issue later on in these notes.

D. $\delta[f(k)]$ for M-shaped Dirac deltas

Let $f(k)$ be a function that vanishes at some k_l , i.e. $f(k_l) = 0$, and let $\delta_n(k)$ be a delta-sequence such that supports of $\delta_n(k - k_1)$ and $\delta_n(k - k_2)$ do not overlap, no matter which k_1 and k_2 , $f(k_1) = f(k_2) = 0$, one takes. I will now show how to derive in the context of M-shaped deltas the analogue of the standard formula

$$\delta[f(k)] = \sum_l \frac{\delta(k - k_l)}{|f'(k_l)|}. \quad (294)$$

Let us first assume that $f'(k_l)$ exists. The first trick is to replace, in a neighborhood of a k_l , the function $f(k)$ by $g_l(k) = f'(k_l)(k - k_l)$ i.e. its tangent line at k_l . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dk \delta_n[f(k)] F(k) &= \sum_l \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dk \delta_n[g_l(k)] F(k) \\ &= \sum_l \lim_{n \rightarrow \infty} \frac{1}{|f'(k_l)|} \int_{-\infty}^{\infty} dk \delta_n(k) F\left(\frac{k + f'(k_l)k_l}{f'(k_l)}\right) \\ &= \sum_l \frac{1}{|f'(k_l)|} \frac{F(k_{l+}) + F(k_{l-})}{2} \end{aligned} \quad (295)$$

which coincides with (294). As an example consider the important case of

$$f(k_0) = k_0^2 - \mathbf{k}^2 - m^2, \quad (296)$$

$$f'(k_0) = 2k_0, \quad (297)$$

$$k_l = \pm \sqrt{\mathbf{k}^2 + m^2}. \quad (298)$$

Let $k_1 = -\sqrt{\mathbf{k}^2 + m^2}$. Then

$$g_1(k_0) = -2\sqrt{\mathbf{k}^2 + m^2}(k_0 + \sqrt{\mathbf{k}^2 + m^2}), \quad (299)$$

$$g_2(k_0) = 2\sqrt{\mathbf{k}^2 + m^2}(k_0 - \sqrt{\mathbf{k}^2 + m^2}). \quad (300)$$

For $m = 0$ the point $\mathbf{k} = 0$ is singular since $f'(k_1) = f'(k_2) = 0$. In expressions such as

$$\phi(x) = \int d^4k \delta(k_0^2 - \mathbf{k}^2) \tilde{\phi}(k_0, \mathbf{k}) e^{-ikx} \quad (301)$$

the function $\tilde{\phi}(k_0, \mathbf{k})$ must satisfy

$$\lim_{k_0 \rightarrow 0_{\pm}} \frac{\tilde{\phi}(k_0, \mathbf{0})}{k_0} = 0. \quad (302)$$

The fact that δ is M-shaped is thus irrelevant in this context.

XI. PLANE WAVES AND M-SHAPED DIRAC DELTAS

In Dirac's bra-ket notation one encounters formulas such as $\langle k|k' \rangle = 2\pi\delta(k - k')$ or $\frac{1}{2\pi} \int_{-\infty}^{\infty} dk |k\rangle \langle k| = 1$. In what follows I want to show in what sense they can be interpreted in the language of M-shaped delta sequences.

Let $\delta_n(k) = \delta_n(k, j)$ for some positive integer j . We start with convolution of two M-shaped delta-sequences,

$$\delta_{nm}^*(k) = \delta_n * \delta_m(k) = \int_{-\infty}^{\infty} dk' \delta_n(k - k') \delta_m(k') = \delta_{mn}^*(k), \quad (303)$$

$$\lim_{m \rightarrow \infty} \delta_{nm}^*(k) = \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} dk' \delta_n(k - k') \delta_m(k') = \delta_n(k), \quad (304)$$

$$\lim_{n \rightarrow \infty} \delta_{nm}^*(k) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dk' \delta_n(k - k') \delta_m(k') = \delta_m(k). \quad (305)$$

The new sequence is again a delta-sequence,

$$\begin{aligned} \int_{-\infty}^{\infty} dk \delta_{nm}^*(k) &= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \delta_n(k - k') \delta_m(k') \\ &= \int_{-\infty}^{\infty} dk' \delta_m(k') \int_{-\infty}^{\infty} dk \delta_n(k - k') \\ &= \int_{-\infty}^{\infty} dk' \delta_m(k') = 1, \end{aligned} \quad (306)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} dk f(k) \delta_{nm}^*(k) &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' f(k) \delta_n(k - k') \delta_m(k') \\ &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dk f(k) \delta_n(k) = \frac{f(0_-) + f(0_+)}{2}, \end{aligned} \quad (307)$$

but

$$\begin{aligned}\delta_{nm}^*(0) &= \int_{-\infty}^{\infty} \delta_n(0 - k') \delta_m(k') dk' \\ &= \int_{-\infty}^{\infty} \delta_n(k') \delta_m(k') dk'\end{aligned}\quad (308)$$

in general depends on n and m . The other properties are nevertheless analogous to M-shaped delta-sequences,

$$\lim_{m \rightarrow \infty} \delta_{nm}^*(0) = \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(k') \delta_m(k') dk' = \delta_n(0) = 0, \quad (309)$$

and

$$\lim_{n \rightarrow \infty} \delta_{nn}^*(0) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(k') \delta_n(k') dk' = \infty.$$

Moreover, there exists a sequence α_{nm} ,

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \alpha_{nm} = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \alpha_{nm} = 0, \quad (310)$$

such that $\delta_{nm}^*(k) = 0$ for $|k| \geq \alpha_{nm}$. Employing the Fourier transform

$$\hat{\delta}_{nm}^*(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \delta_{nm}^*(k) e^{ikx} = 2\pi \hat{\delta}_n(x) \hat{\delta}_m(x) \quad (311)$$

we can write

$$\begin{aligned}\delta_{nm}^*(k - k') &= \int_{-\infty}^{\infty} dx \hat{\delta}_{nm}^*(x) e^{-i(k-k')x} \\ &= 2\pi \int_{-\infty}^{\infty} dx \overline{\hat{\delta}_n(x)} e^{ikx} \hat{\delta}_m(x) e^{ik'x}\end{aligned}\quad (312)$$

since $\hat{\delta}_n(x)$ is real. The evaluation map $\langle x|\psi\rangle = \hat{\psi}(x)$, returning the value of Fourier transform of a given ψ , can be used to denote

$$\langle x|k, n\rangle = 2\pi \hat{\delta}_n(x) e^{ikx} = 2\pi \hat{\delta}_n(x) \langle x|k\rangle, \quad (313)$$

$$\lim_{n \rightarrow \infty} \langle x|k, n\rangle = e^{ikx} =: \langle x|k\rangle, \quad (314)$$

$$\langle k, n|x\rangle = \overline{\langle x|k, n\rangle} = 2\pi \hat{\delta}_n(x) e^{-ikx} = 2\pi \hat{\delta}_n(x) \langle k|x\rangle, \quad (315)$$

$$\lim_{n \rightarrow \infty} \langle k, n|x\rangle = e^{-ikx} =: \langle k|x\rangle. \quad (316)$$

Accordingly

$$\delta_{nm}^*(k - k') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \langle k, n|x\rangle \langle x|k', m\rangle \quad (317)$$

$$= \frac{1}{2\pi} \langle k, n|k', m\rangle. \quad (318)$$

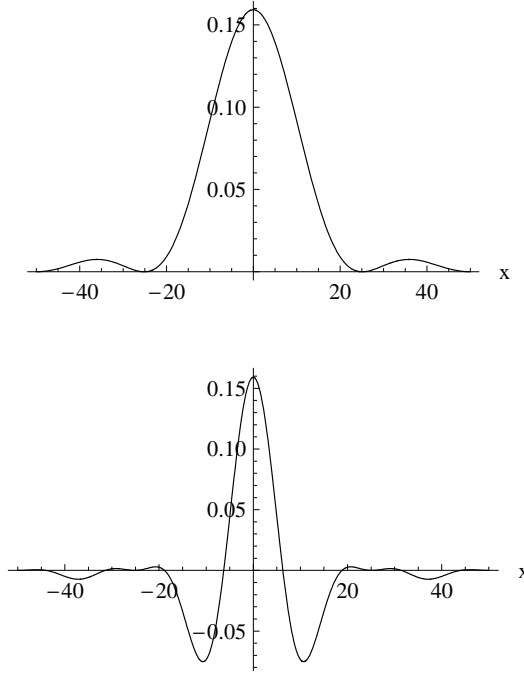


FIG. 5: Plots of (326) for $n = 4$ and different values of j : $j = 0$ (upper) and $j = 1$ (lower). For $j = 1$ the integral (327) vanishes due to the oscillating term, whereas the integrand in (328) is non-negative.

The fact that $\hat{\delta}_n(x)$ is a Fourier transform of a square-integrable $\delta_n(k)$ implies, by Plancherel's theorem [39], that $\langle x|k, n\rangle$ is square integrable for finite n ,

$$\int_{-\infty}^{\infty} dx |\langle x|k, n\rangle|^2 = \int_{-\infty}^{\infty} dx |2\pi \hat{\delta}_n(x) e^{ikx}|^2 = (2\pi)^2 \int_{-\infty}^{\infty} dx |\hat{\delta}_n(x)|^2 < \infty. \quad (319)$$

A similar result holds for

$$|\delta_{nm}^*(k - k')|^2 = \frac{1}{(2\pi)^2} |\langle k, n|k', m\rangle|^2 \quad (320)$$

$$\leq \frac{1}{(2\pi)^2} \langle k, n|k, n\rangle \langle k', m|k', m\rangle < \infty. \quad (321)$$

Let us finally return to (312) and take the (well defined) limit,

$$\begin{aligned} \lim_{m \rightarrow \infty} \delta_{nm}^*(k - k') &= 2\pi \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} dx \overline{\hat{\delta}_n(x) e^{ikx}} \hat{\delta}_m(x) e^{ik'x}, \\ &= \int_{-\infty}^{\infty} dx \hat{\delta}_n(x) e^{-i(k-k')x} = \delta_n(k - k'), \end{aligned} \quad (322)$$

which is just the inverse Fourier transform of a square integrable function. The next limit

would be, however, purely formal

$$\lim_{n \rightarrow \infty} \delta_n(k - k') \stackrel{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-i(k-k')x} \stackrel{?}{=} \delta(k - k') \stackrel{?}{=} \delta^*(k - k'), \quad (323)$$

still showing that the familiar expression for the Fourier form of Dirac's delta is true also for deltas constructed by means M-shaped delta sequences vanishing at 0. But, then, what about the celebrated divergency

$$\delta(0) \stackrel{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \stackrel{?}{=} \infty? \quad (324)$$

Let us do the calculation more precisely,

$$\lim_{m \rightarrow \infty} \delta_{nm}^*(0) = \int_{-\infty}^{\infty} dx \hat{\delta}_n(x) = \delta_n(0). \quad (325)$$

Inserting our explicit example,

$$\hat{\delta}_n(x) = \frac{1}{2\pi} \left(\frac{x}{2n}\right)^{-2} \sin^2 \frac{x}{2n} \cos \frac{jx}{n}, \quad (326)$$

we find that the integral vanishes for M-shaped delta sequences ($j = 1, 2, \dots$)

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left(\frac{x}{2n}\right)^{-2} \sin^2 \frac{x}{2n} \cos \frac{jx}{n} = 0, \quad (327)$$

and for the Λ -shaped one ($j = 0$),

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left(\frac{x}{2n}\right)^{-2} \sin^2 \frac{x}{2n} = n, \quad (328)$$

diverges in the limit $n \rightarrow \infty$, as expected (compare Fig. 5).

XII. M-SHAPED DIRAC DELTAS AND SPECTRAL THEOREM

Following Dirac, we will denote scalar products of square-integrable functions by the same bra-ket symbol as the evaluation map. Let $\hat{\psi}(x) = \langle x | \psi \rangle$ be square-integrable. Then

$$\begin{aligned} \langle k, n | \psi \rangle &= \int_{-\infty}^{\infty} dx \langle k, n | x \rangle \langle x | \psi \rangle \\ &= \int_{-\infty}^{\infty} dx 2\pi \hat{\delta}_n(x) e^{-ikx} \hat{\psi}(x). \end{aligned} \quad (329)$$

(329) is a kind of windowed Fourier transform [39] of $\psi(x)$, with the window function $2\pi \hat{\delta}_n(x)$.

Let us again be as explicit as possible and employ the concrete Λ -shaped sequence (290),

$$\begin{aligned}
\langle k, n | \psi \rangle &= \int_{-\infty}^{\infty} dx \, 2\pi \frac{1}{2\pi} \left(\frac{x}{2n} \right)^{-2} \sin^2 \frac{x}{2n} \cos \frac{jx}{n} e^{-ikx} \hat{\psi}(x) \\
&= \int_{-\infty}^{\infty} dx \, e^{-ikx} \underbrace{\left(\frac{x}{2n} \right)^{-2} \sin^2 \frac{x}{2n} \cos \frac{jx}{n}}_{\hat{\psi}_n(x, j)} \hat{\psi}(x) \\
&= \int_{-\infty}^{\infty} dx \, e^{-ikx} \hat{\psi}_n(x, j).
\end{aligned} \tag{330}$$

The Parseval theorem,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \langle \psi | k, m \rangle \langle k, n | \psi \rangle = \int_{-\infty}^{\infty} dx \, \overline{\hat{\psi}_m(x, j)} \hat{\psi}_n(x, j), \tag{331}$$

with

$$|\overline{\hat{\psi}_m(x, j)} \hat{\psi}_n(x, j)| = |\hat{\psi}(x)|^2 \underbrace{\left| \left(\frac{x}{2m} \right)^{-2} \sin^2 \frac{x}{2m} \cos \frac{jx}{m} \left(\frac{x}{2n} \right)^{-2} \sin^2 \frac{x}{2n} \cos \frac{jx}{n} \right|}_{\leq 1} \leq |\hat{\psi}(x)|^2,$$

allows us to use the dominated convergence theorem (Theorem 6 in [40]) and take the limits, in any order, under the integral sign,

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \langle \psi | k, m \rangle \langle k, n | \psi \rangle \\
&= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} dx \, |\hat{\psi}(x)|^2 \left(\frac{x}{2m} \right)^{-2} \left(\frac{x}{2n} \right)^{-2} \sin^2 \frac{x}{2m} \cos \frac{jx}{m} \sin^2 \frac{x}{2n} \cos \frac{jx}{n} \\
&= \int_{-\infty}^{\infty} dx \, |\hat{\psi}(x)|^2 \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \left(\frac{x}{2m} \right)^{-2} \left(\frac{x}{2n} \right)^{-2} \sin^2 \frac{x}{2m} \cos \frac{jx}{m} \sin^2 \frac{x}{2n} \cos \frac{jx}{n} \\
&= \int_{-\infty}^{\infty} dx \, |\hat{\psi}(x)|^2.
\end{aligned} \tag{332}$$

The same argument can be applied to the diagonal limit ($n = m$),

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \langle \psi | k, n \rangle \langle k, n | \psi \rangle &= \int_{-\infty}^{\infty} dx \, |\hat{\psi}(x)|^2 \lim_{n \rightarrow \infty} \left| \left(\frac{x}{2n} \right)^{-2} \sin^2 \frac{x}{2n} \cos \frac{jx}{n} \right|^2 \\
&= \int_{-\infty}^{\infty} dx \, |\hat{\psi}(x)|^2 = \langle \psi | \psi \rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \lim_{n \rightarrow \infty} \langle \psi | k, n \rangle \langle k, n | \psi \rangle.
\end{aligned}$$

Expressing scalar products in terms of norms (the polarization identity [39]),

$$\langle f | g \rangle = \frac{1}{4} \left(\|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - i\|f - ig\|^2 \right), \tag{333}$$

we can extend the result to pairs of arbitrary square-integrable functions ψ and ϕ ,

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \langle \phi | k, n \rangle \langle k, n | \psi \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \lim_{n \rightarrow \infty} \langle \phi | k, n \rangle \langle k, n | \psi \rangle \\
&= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \langle \phi | k, m \rangle \langle k, n | \psi \rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle \phi | k, m \rangle \langle k, n | \psi \rangle \\
&= \langle \phi | \psi \rangle.
\end{aligned} \tag{334}$$

In this sense

$$\begin{aligned}
\mathbb{I} &= \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk |k, n\rangle \langle k, n| \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \lim_{n \rightarrow \infty} |k, n\rangle \langle k, n| \\
&= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk |k, m\rangle \langle k, n| \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} |k, m\rangle \langle k, n| \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk |k\rangle \langle k|.
\end{aligned} \tag{335}$$

Now consider the matrix element

$$\begin{aligned}
\langle \phi | A | \psi \rangle &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) \langle \phi | k, m \rangle \langle k, n | \psi \rangle \\
&= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) \int_{-\infty}^{\infty} dx \langle \hat{\phi} | x \rangle \langle x | k, m \rangle \int_{-\infty}^{\infty} dy \langle k, n | y \rangle \langle y | \hat{\psi} \rangle \\
&= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) \int_{-\infty}^{\infty} dx \overline{\hat{\phi}(x)} 2\pi \hat{\delta}_m(x) e^{ikx} \int_{-\infty}^{\infty} dy 2\pi \hat{\delta}_n(y) \hat{\psi}(y) e^{-iky} \\
&= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) \int_{-\infty}^{\infty} dx 2\pi \hat{\delta}_m(x) \hat{\phi}(x) e^{-ikx} \int_{-\infty}^{\infty} dy 2\pi \hat{\delta}_n(y) \hat{\psi}(y) e^{-iky}.
\end{aligned}$$

With our definitions of Fourier transforms,

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk f(k) e^{ikx}, \tag{336}$$

$$f(k) = \int_{-\infty}^{\infty} dx \hat{f}(x) e^{-ikx}, \tag{337}$$

$$\int_{-\infty}^{\infty} dx \hat{f}(x) \hat{g}(x) e^{-ikx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk' f(k - k') g(k'), \tag{338}$$

we obtain

$$\langle \phi | A | \psi \rangle = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) \underbrace{\int_{-\infty}^{\infty} dk' \delta_m(k - k') \overline{\phi(k')}}_{\delta_m * \overline{\phi(k)}} \underbrace{\int_{-\infty}^{\infty} dk'' \delta_n(k - k'') \psi(k'')}_{\delta_n * \psi(k)}.$$

As a by-product of the above calculation we have shown that $\langle k, n | \psi \rangle = \delta_n * \psi(k)$.

Of great importance is the following

Lemma: (a variant of Theorem II.3.1.1 in [31]) If $\psi(k)$ is continuous then $\langle k, n | \psi \rangle = \delta_n * \psi(k)$ converges to $\psi(k)$ almost uniformly.

Proof: Assume for simplicity that the delta-sequence is non-negative, $\delta_n(k) \geq 0$, and vanishes for $|k| > \alpha_n$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, like in all our previous examples. Assume k belongs to a finite interval $[a, b]$. Then

$$\begin{aligned} |\delta_n * \psi(k) - \psi(k)| &= \left| \int_{-\infty}^{\infty} dk' \delta_n(k - k') \psi(k') - \psi(k) \int_{-\infty}^{\infty} dk' \delta_n(k - k') \right| \\ &= \left| \int_{-\infty}^{\infty} dk' \delta_n(k - k') (\psi(k') - \psi(k)) \right| \\ &\leq \int_{-\infty}^{\infty} dk' \delta_n(k - k') |\psi(k') - \psi(k)| \\ &= \int_{-\alpha_n}^{\alpha_n} dk' \delta_n(k') |\psi(k - k') - \psi(k)| \end{aligned}$$

By definition of continuity of $\psi(k)$ at k , for any $\varepsilon > 0$ there exists n_0 such that $|\psi(k - k') - \psi(k)| < \varepsilon$ for all $n > n_0$ and $-\alpha_n \leq k' \leq \alpha_n$. So

$$\begin{aligned} |\delta_n * \psi(k) - \psi(k)| &\leq \int_{-\alpha_n}^{\alpha_n} dk' \delta_n(k') |\psi(k - k') - \psi(k)| \\ &\leq \varepsilon \int_{-\alpha_n}^{\alpha_n} dk' \delta_n(k') = \varepsilon \end{aligned}$$

for all $n > n_0$ and $a \leq k \leq b$. ■

We can take the limits under the integral sign in

$$\langle \phi | A | \psi \rangle = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) \delta_m * \overline{\phi(k)} \delta_n * \psi(k)$$

in several cases. The simplest is the one of $A(k) \delta_m * \overline{\phi(k)} \delta_n * \psi(k)$ converging uniformly to $A(k) \delta_m * \overline{\phi(k)} \psi(k)$ and $A(k) \overline{\phi(k)} \delta_n * \psi(k)$. For, example let $A(k)$ be continuous for $a < k < b$, and $A(k) = 0$ for $k \geq b$, $k \leq a$. Almost uniform convergence implies uniform convergence

on any closed interval, and

$$\begin{aligned}
\langle \phi | A | \psi \rangle &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_a^b dk A(k) \delta_m * \overline{\phi(k)} \delta_n * \psi(k) \\
&= \frac{1}{2\pi} \int_a^b dk A(k) \lim_{m \rightarrow \infty} \delta_m * \overline{\phi(k)} \lim_{n \rightarrow \infty} \delta_n * \psi(k) \\
&= \frac{1}{2\pi} \int_a^b dk A(k) \overline{\phi(k)} \psi(k) \\
&= \frac{1}{2\pi} \int_a^b dk A(k) \langle \phi | k \rangle \langle k | \psi \rangle.
\end{aligned}$$

A generalization going beyond finite intervals $a \leq k \leq b$ can be based on the observation that dominated convergence theorem guarantees

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) \delta_m * \overline{\phi(k)} \delta_n * \psi(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) \overline{\phi(k)} \psi(k) \quad (339)$$

if the right-hand-side of (339) is finite and

$$|A(k) \delta_m * \overline{\phi(k)} \delta_n * \psi(k)| \leq \text{const}_1 \times |A(k) \delta_m * \overline{\phi(k)} \psi(k)| \leq \text{const}_2 \times |A(k) \overline{\phi(k)} \psi(k)|.$$

Let us now turn to the issue of eigenvalues of A . Let $\phi(k)$ be continuous. The eigenvalue problem for A should be understood in this formalism in the following sense

$$\begin{aligned}
\langle \phi | A | p \rangle &= \lim_{l \rightarrow \infty} \langle \phi | A | p, l \rangle \\
&= \lim_{l \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) \langle \phi | k, m \rangle \langle k, n | p, l \rangle \\
&= \lim_{l \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dk A(k) \langle \phi | k, m \rangle \delta_{nl}(k - p) \quad (340)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{m \rightarrow \infty} \frac{1}{2} \left(A(p_-) + A(p_+) \right) \langle \phi | p, m \rangle \\
&= \frac{1}{2} \left(A(p_-) + A(p_+) \right) \langle \phi | p \rangle. \quad (341)
\end{aligned}$$

Let us recall that $\langle \phi | k, m \rangle$ converges almost uniformly to a continuous $\langle \phi | k \rangle = \overline{\phi(k)}$, while the support of $\delta_{nl}(k - p)$ is compact. Therefore, the sequence $A(k) \langle \phi | k, m \rangle \delta_{nl}(k - p)$ converges uniformly to $A(k) \langle \phi | k \rangle \delta_{nl}(k - p)$ if $A(k)$ is bounded on any compact subset of \mathbb{R} . In consequence, one can alternatively compute the eigenvalue problem as follows [compare (340)]

$$\langle \phi | A | p \rangle = \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} dk A(k) \langle \phi | k \rangle \delta_{nl}(k - p) \quad (342)$$

$$= \frac{1}{2} \left(A(p_-) + A(p_+) \right) \langle \phi | p \rangle. \quad (343)$$

If continuous $\phi(k)$ is itself of compact support then $\langle\phi|k, m\rangle$ converges to $\langle\phi|k\rangle$ uniformly.

Although (341) shows that the sequential approach automatically defines eigenvalues also at points of discontinuity, in what follows I simplify discussion and assume that $A(k)$ is continuous. In such a case the formalism based on M-shaped deltas does not seem to essentially differ from other mathematical approaches to the Dirac bra-ket formalism for continuous spectra (cf. [41–43]). Anyway, what we modify is the value of $\langle k|k'\rangle$ on the diagonal $k = k'$. Looking at the rigged Hilbert space approach, say, one does not find any point where the formalism employs a concrete value (infinite or not) of $\langle k|k\rangle$.

The main difference with respect to more standard formalisms thus lies in definition of diagonal elements of operators whose spectrum is continuous. Indeed,

$$\begin{aligned}
\langle p|A|p\rangle &= \lim_{r\rightarrow\infty} \lim_{s\rightarrow\infty} \langle p, r|A|p, s\rangle \\
&= \lim_{r\rightarrow\infty} \lim_{s\rightarrow\infty} \lim_{m\rightarrow\infty} \lim_{n\rightarrow\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) \langle p, r|k, m\rangle \langle k, n|p, s\rangle \\
&= \lim_{r\rightarrow\infty} \lim_{s\rightarrow\infty} \lim_{m\rightarrow\infty} \lim_{n\rightarrow\infty} 2\pi \int_{-\infty}^{\infty} dk A(k) \delta_{rm}(p-k) \delta_{ns}(k-p) \\
&= \lim_{s\rightarrow\infty} \lim_{n\rightarrow\infty} 2\pi A(p) \delta_{ns}(0) = 0.
\end{aligned} \tag{344}$$

In particular, $\langle k|k\rangle = 0$. The latter, of course, does not mean that $|k\rangle = \lim_{n\rightarrow\infty} |k, n\rangle$ is vanishing. Simply, as has been stressed here many times, neither $\langle k|k'\rangle$ nor $\langle k|\psi\rangle$ should be regarded as scalar products.

It follows that $|k\rangle\langle k|$ is not a projector,

$$|k\rangle\langle k|k\rangle\langle k| = \delta(0)|k\rangle\langle k| = 0. \tag{345}$$

However, let $\chi_X(k)$ be the characteristic function of $X \subset \mathbb{R}$, i.e. $\chi_X(k) = 1$ if $k \in X$, and $\chi_X(k) = 0$ if $k \notin X$. The operators

$$E(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \chi_X(k) |k\rangle\langle k|, \tag{346}$$

$$E([a, b]) = \frac{1}{2\pi} \int_a^b dk |k\rangle\langle k|, \tag{347}$$

are projectors (\emptyset is the empty set):

$$E(X)E(Y) = E(X \cap Y), \quad (348)$$

$$E(\emptyset) = 0, \quad (349)$$

$$E(\mathbb{R}) = \mathbb{I}, \quad (350)$$

$$E([a, b]) \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) |k\rangle\langle k| = \frac{1}{2\pi} \int_a^b dk A(k) |k\rangle\langle k|. \quad (351)$$

The operator $|k\rangle\langle k|$ is not a projector but a POVM [17], since

$$\begin{aligned} \langle\psi|(|k\rangle\langle k|)|\psi\rangle &= \lim_{n_1 \rightarrow \infty} \dots \lim_{n_4 \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \overline{\psi(p)} \langle p, n_1 | k, n_2 \rangle \langle k, n_3 | \frac{1}{2\pi} \int_{-\infty}^{\infty} dp' \psi(p') | p', n_4 \rangle \\ &= \lim_{n_1 \rightarrow \infty} \dots \lim_{n_4 \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \frac{1}{2\pi} \int_{-\infty}^{\infty} dp' \overline{\psi(p)} \psi(p') \langle p, n_1 | k, n_2 \rangle \langle k, n_3 | p', n_4 \rangle \\ &= \lim_{n_1 \rightarrow \infty} \dots \lim_{n_4 \rightarrow \infty} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \overline{\psi(p)} \psi(p') \delta_{n_1 n_2}(p - k) \delta_{n_3 n_4}(k - p') \\ &= \langle\psi|k\rangle\langle k|\psi\rangle = |\psi(k)|^2 \geq 0, \end{aligned} \quad (352)$$

and $\mathbb{I} = \int dk/(2\pi) |k\rangle\langle k|$.

Remark: The above calculations show what should be meant by products of operators. For example, let A, B, C commute. Then

$$\begin{aligned} ABC &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) |k\rangle\langle k| \int_{-\infty}^{\infty} \frac{dk'}{2\pi} B(k') |k'\rangle\langle k'| \int_{-\infty}^{\infty} \frac{dk''}{2\pi} C(k'') |k''\rangle\langle k''| \\ &= \lim_{n_1 \rightarrow \infty} \dots \lim_{n_2'' \rightarrow \infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) |k, n_1\rangle\langle k, n_2| \int_{-\infty}^{\infty} \frac{dk'}{2\pi} B(k') |k', n_1'\rangle\langle k', n_2'| \\ &\quad \times \int_{-\infty}^{\infty} \frac{dk''}{2\pi} C(k'') |k'', n_1''\rangle\langle k'', n_2''| \\ &= \lim_{n_1 \rightarrow \infty} \dots \lim_{n_2'' \rightarrow \infty} \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} A(k) B(k') C(k'') |k, n_1\rangle\langle k, n_2| k', n_1'\rangle\langle k', n_2'| k'', n_1''\rangle\langle k'', n_2''| \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) B(k) C(k) |k\rangle\langle k| \end{aligned} \quad (353)$$

independently of the order of limits, the latter property being equivalent to associativity of the product.▲

As we can see, one can work with M-shaped deltas with no difficulty. Restricting the class of Mikusiński deltas to a subset of $[\delta_n]$ we obtain the usual Dirac deltas but equipped with new and useful properties. The fact that we work with kets of “zero length” is no more paradoxical than what we face when we deal with those of “infinite length”.

Paraphrasing physics textbook presentations of $\delta(x)$ we could say that M-shaped delta is a “function which is zero everywhere, but whose integral over any interval containing 0 equals unity”. Such a “definition” is neither worse nor better than the usual one with infinity at 0 — the difference between them is on the set of zero measure!

Let me end this section with remarks on relativistic M-shaped Dirac deltas. In what follows two types of expressions involving Dirac deltas will be encountered. First,

$$\delta^{*(3)}(\mathbf{k}) = \delta^*(k_1)\delta^*(k_2)\delta^*(k_3) = [\delta_{m_1 n_1}^*(k_1)][\delta_{m_2 n_2}^*(k_2)][\delta_{m_3 n_3}^*(k_3)], \quad (354)$$

$$1 = \int_{\mathbb{R}^3} d^3k \delta^{*(3)}(\mathbf{k}), \quad (355)$$

is M-shaped in 1D and 3D: $\delta^*(0) = 0$, $\delta^{*(3)}(\mathbf{0}) = 0$. Sometimes it may be useful to employ the integral formula (212) for $\delta^{(3)}(\mathbf{p})$ (under appropriate integrals) but then $\delta^{(3)}(\mathbf{0})$ is meaningless. From now on I will stick to the convention that \mathbf{p} denotes the spacelike part of the 4-vector $p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p}) = (p_0, \mathbf{p})$, whereas \mathbf{k} stands for an analogous part of the null 4-vector $k = (|\mathbf{k}|, \mathbf{k}) = (k_0, \mathbf{k})$. Relativistically covariant kets will be normalized as follows,

$$\langle \mathbf{p} | \mathbf{p}' \rangle = (2\pi)^3 2\sqrt{\mathbf{p}^2 + m^2} \delta^{*(3)}(\mathbf{p} - \mathbf{p}') =: \delta_m(\mathbf{p}, \mathbf{p}'), \quad (356)$$

$$\langle \mathbf{k} | \mathbf{k}' \rangle = (2\pi)^3 2|\mathbf{k}| \delta^{*(3)}(\mathbf{k} - \mathbf{k}') =: \delta_0(\mathbf{k}, \mathbf{k}'), \quad (357)$$

$$\delta_m(\mathbf{p}, \mathbf{p}) = \delta_0(\mathbf{k}, \mathbf{k}) = 0. \quad (358)$$

The resolutions of unity are

$$\int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} |\mathbf{k}\rangle \langle \mathbf{k}| = \int_{\mathbb{R}^3} dk |\mathbf{k}\rangle \langle \mathbf{k}| = \mathbb{I}, \quad (359)$$

$$\int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3 2\sqrt{\mathbf{p}^2 + m^2}} |\mathbf{p}\rangle \langle \mathbf{p}| = \int_{\mathbb{R}^3} dp |\mathbf{p}\rangle \langle \mathbf{p}| = \mathbb{I}. \quad (360)$$

Here dp and dk denote relativistic measures on the hyperboloid $p^2 = m^2$ and the light cone $k^2 = 0$, respectively.

XIII. ALGEBRA OF FREE FIELD OPERATORS REVISITED

The ultimate goal of these lecture notes is to discuss a theory of interacting quantum relativistic scalar fields: A charged massive boson $\psi(x)$ and a neutral massless boson $\phi(x)$.

For free fields one finds

$$\psi(x) = \int dp \left(a(\mathbf{p}) e^{-ipx} + b(\mathbf{p})^\dagger e^{ipx} \right), \quad (361)$$

$$\phi(x) = \int dk \left(c(\mathbf{k}) e^{-ikx} + c(\mathbf{k})^\dagger e^{ikx} \right). \quad (362)$$

The amplitude particle and antiparticle operators are assumed to satisfy bosonic HOLAs:

$$[a(\mathbf{p}), a(\mathbf{p}')^\dagger] = \delta_m(\mathbf{p}, \mathbf{p}') I_m(\mathbf{p}), \quad (363)$$

$$[a(\mathbf{p}), n_+(\mathbf{p}')] = \delta_m(\mathbf{p}, \mathbf{p}') a(\mathbf{p}), \quad (364)$$

$$[a(\mathbf{p})^\dagger, n_+(\mathbf{p}')] = -\delta_m(\mathbf{p}, \mathbf{p}') a(\mathbf{p})^\dagger, \quad (365)$$

$$[b(\mathbf{p}), b(\mathbf{p}')^\dagger] = \delta_m(\mathbf{p}, \mathbf{p}') I_m(\mathbf{p}), \quad (366)$$

$$[b(\mathbf{p}), n_-(\mathbf{p}')] = \delta_m(\mathbf{p}, \mathbf{p}') b(\mathbf{p}), \quad (367)$$

$$[b(\mathbf{p})^\dagger, n_-(\mathbf{p}')] = -\delta_m(\mathbf{p}, \mathbf{p}') b(\mathbf{p})^\dagger. \quad (368)$$

The massless neutral field satisfies a similar algebra but with the light cone deltas,

$$[c(\mathbf{k}), c(\mathbf{k}')^\dagger] = \delta_0(\mathbf{k}, \mathbf{k}') I_0(\mathbf{k}), \quad (369)$$

$$[c(\mathbf{k}), n_0(\mathbf{k}')] = \delta_0(\mathbf{k}, \mathbf{k}') c(\mathbf{k}), \quad (370)$$

$$[c(\mathbf{k})^\dagger, n_0(\mathbf{k}')] = -\delta_0(\mathbf{k}, \mathbf{k}') c(\mathbf{k})^\dagger. \quad (371)$$

The remaining commutators vanish.

Remark: Let us immediately note the peculiarity of M-shaped deltas:

$$[a(\mathbf{p}), a(\mathbf{p})^\dagger] = [b(\mathbf{p}), b(\mathbf{p})^\dagger] = [c(\mathbf{k}), c(\mathbf{k})^\dagger] = 0. \quad (372)$$

This type of commutation relations is, in fact, implicitly employed in standard quantum field theory whenever one applies the normal ordering operation ($: \hat{q} :$ instead of \hat{q} , etc.). I think this is yet another argument for quantization in terms of M-shaped deltas.▲

Remark: Let us divide \mathbb{R}^3 into disjointed sets X_n , $\bigcup_{n=0}^\infty X_n = \mathbb{R}^3$, and let $\chi_n(\mathbf{p})$ be their characteristic functions satisfying $\sum_{n=0}^\infty \chi_n(\mathbf{p}) = 1$ and $\chi_n(\mathbf{p}) \chi_j(\mathbf{p}) = \chi_n(\mathbf{p}) \delta_{nj}$. Here δ_{nj} is the Kronecker delta. Defining

$$a_n = \int dp \chi_n(\mathbf{p}) a(\mathbf{p}) \quad (373)$$

we find that

$$\begin{aligned}
[a_n, a_n^\dagger] &= \left[\int dp \chi_n(\mathbf{p}) a(\mathbf{p}), \int dp' \chi_n(\mathbf{p}') a(\mathbf{p}')^\dagger \right] \\
&= \int dp \int dp' \chi_n(\mathbf{p}) \chi_n(\mathbf{p}') [a(\mathbf{p}), a(\mathbf{p}')^\dagger] \\
&= \int dp \int dp' \chi_n(\mathbf{p}) \chi_n(\mathbf{p}') \delta_m(\mathbf{p}, \mathbf{p}') I_m(\mathbf{p}) \\
&= \int dp \chi_n(\mathbf{p})^2 I_m(\mathbf{p}) \\
&= \int dp \chi_n(\mathbf{p}) I_m(\mathbf{p}) = I_{m,n} \neq 0.
\end{aligned} \tag{374}$$

The example shows that commutability (372) typical of M-shaped deltas does *not* imply that a discrete version of the representation should be automatically taken in the form $[a_n, a_n^\dagger] = 0$. The discrete counterpart of HOLA will thus read

$$[a_n, a_j^\dagger] = I_n \delta_{nj} \tag{375}$$

with appropriately chosen I_n . \blacktriangle

The above fields will be used by me in two nontrivial toy models. The first model involves interaction term (in interaction picture)

$$H_1(t) = \int_{\mathbb{R}^3} d^3x j_0(t, \mathbf{x}) \phi(t, \mathbf{x}) \tag{376}$$

$$= iq \int_{\mathbb{R}^3} d^3x \left(\psi(t, \mathbf{x}) \dot{\psi}(t, \mathbf{x}) - \dot{\psi}(t, \mathbf{x}) \psi(t, \mathbf{x}) \right) \phi(t, \mathbf{x}) \tag{377}$$

$$= \int_{\mathbb{R}^3} d^3x \hat{\rho}(t, \mathbf{x}) \phi(t, \mathbf{x}), \tag{378}$$

where

$$j_\mu(x) = iq\psi(x)\partial_\mu\psi(x) - iq\partial_\mu\psi(x)\psi(x) \tag{379}$$

is the charge-density current of the massive field, and $\hat{\rho}(t, \mathbf{x})$ is the charge-density operator,

$$\int_{\mathbb{R}^3} d^3x \hat{\rho}(t, \mathbf{x}) = \int_{\mathbb{R}^3} d^3x \hat{\rho}(0, \mathbf{x}) = \hat{q}[\psi^\dagger, \psi]. \tag{380}$$

The fields that define the Hamiltonian are *free*, a fact typical of the interaction picture. Note that I do not assume normal ordering of $\hat{\rho}(t, \mathbf{x})$.

The Hamiltonian has the essential properties of quantum-electrodynamical interaction, but is not relativistically covariant. For this reason I will also consider a more complicated

interaction, with the same charge current,

$$H_1(t) = \int_{\Sigma} d\sigma^{\mu}(x) j_{\mu}(t, \mathbf{x}) \phi(t, \mathbf{x}), \quad (381)$$

where Σ is a hypersurface in Minkowski space and $d\sigma^{\mu}(x)$ a measure multiplied by a vector field of vectors normal to Σ . Analysis of (at least) loop diagrams for the latter interaction is the principal goal of these lecture notes.

A. Reducible $N \geq 1$ representation of HOLA — an indefinite-frequency harmonic oscillator representation of $\psi(x)$

Let $[\hat{a}(\mathbf{p}), \hat{a}(\mathbf{p}')^{\dagger}] = \iota_{+}(\mathbf{p}, \mathbf{p}')$, $[\hat{b}(\mathbf{p}), \hat{b}(\mathbf{p}')^{\dagger}] = \iota_{-}(\mathbf{p}, \mathbf{p}')$, where ι_{\pm} are any functions (or distributions) satisfying $\iota_{\pm}(\mathbf{p}, \mathbf{p}) = 1$. We assume that all commutators between the a s and the b s vanish. Then $N = 1$ reducible representation is constructed analogously to the indefinite-frequency harmonic oscillator discussed earlier in these notes. The particle part,

$$a(\mathbf{p}, 1) = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes \hat{a}(\mathbf{p}), \quad (382)$$

$$a(\mathbf{p}, 1)^{\dagger} = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes \hat{a}(\mathbf{p})^{\dagger}, \quad (383)$$

$$n_{+}(\mathbf{p}, 1) = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes \hat{a}(\mathbf{p})^{\dagger} \hat{a}(\mathbf{p}), \quad (384)$$

$$I_m(\mathbf{p}, 1) = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes 1, \quad (385)$$

$$\int d\mathbf{p} I_m(\mathbf{p}, 1) = I_m(1), \quad (\text{resolution of identity}) \quad (386)$$

and the antiparticle part,

$$b(\mathbf{p}, 1) = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes \hat{b}(\mathbf{p}), \quad (387)$$

$$b(\mathbf{p}, 1)^{\dagger} = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes \hat{b}(\mathbf{p})^{\dagger}, \quad (388)$$

$$n_{-}(\mathbf{p}, 1) = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes \hat{b}(\mathbf{p})^{\dagger} \hat{b}(\mathbf{p}), \quad (389)$$

$$I_m(\mathbf{p}, 1) = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes 1, \quad (390)$$

can be explicitly constructed in various ways. The simplest case occurs if $\hat{a}(\mathbf{p})$, $\hat{b}(\mathbf{p})$ are \mathbf{p} -independent, i.e.

$$a(\mathbf{p}, 1) = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes (\hat{a} \otimes 1), \quad (391)$$

$$a(\mathbf{p}, 1)^{\dagger} = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes (\hat{a}^{\dagger} \otimes 1), \quad (392)$$

$$I_m(\mathbf{p}, 1) = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes (1 \otimes 1), \quad (393)$$

$$n_{+}(\mathbf{p}, 1) = |\mathbf{p}\rangle \langle \mathbf{p}| \otimes (\hat{a}^{\dagger} \hat{a} \otimes 1), \quad (394)$$

$$b(\mathbf{p}, 1) = |\mathbf{p}\rangle\langle\mathbf{p}| \otimes (1 \otimes \hat{a}), \quad (395)$$

$$b(\mathbf{p}, 1)^\dagger = |\mathbf{p}\rangle\langle\mathbf{p}| \otimes (1 \otimes \hat{a}^\dagger), \quad (396)$$

$$I_m(\mathbf{p}, 1) = |\mathbf{p}\rangle\langle\mathbf{p}| \otimes (1 \otimes 1), \quad (397)$$

$$n_-(\mathbf{p}, 1) = |\mathbf{p}\rangle\langle\mathbf{p}| \otimes (1 \otimes \hat{a}^\dagger \hat{a}), \quad (398)$$

with $[\hat{a}, \hat{a}^\dagger] = 1$. At the other extreme is $\iota_\pm(\mathbf{p}, \mathbf{p}')$ that equals an M-shaped Dirac delta normalized to 1 at $\mathbf{0}$. Both cases are worthy of consideration.

For $N > 1$

$$a(\mathbf{p}, N) = \frac{1}{\sqrt{N}} \left(a(\mathbf{p}, 1) \otimes I_m(1) \otimes \cdots \otimes I_m(1) + \cdots + I_m(1) \otimes \cdots \otimes I_m(1) \otimes a(\mathbf{p}, 1) \right), \quad (399)$$

$$a(\mathbf{p}, N)^\dagger = \frac{1}{\sqrt{N}} \left(a(\mathbf{p}, 1)^\dagger \otimes I_m(1) \otimes \cdots \otimes I_m(1) + \cdots + I_m(1) \otimes \cdots \otimes I_m(1) \otimes a(\mathbf{p}, 1)^\dagger \right), \quad (400)$$

$$n_+(\mathbf{p}, N) = n_+(\mathbf{p}, 1) \otimes I_m(1) \otimes \cdots \otimes I_m(1) + \cdots + I_m(1) \otimes \cdots \otimes I_m(1) \otimes n_+(\mathbf{p}, 1), \quad (401)$$

$$I_m(\mathbf{p}, N) = \frac{1}{N} \left(I_m(\mathbf{p}, 1) \otimes I_m(1) \otimes \cdots \otimes I_m(1) + \cdots + I_m(1) \otimes \cdots \otimes I_m(1) \otimes I_m(\mathbf{p}, 1) \right), \quad (402)$$

$$I_m(N) = \int d\mathbf{p} I_m(\mathbf{p}, N), \quad (\text{resolution of identity}) \quad (403)$$

$$b(\mathbf{p}, N) = \frac{1}{\sqrt{N}} \left(b(\mathbf{p}, 1) \otimes I_m(1) \otimes \cdots \otimes I_m(1) + \cdots + I_m(1) \otimes \cdots \otimes I_m(1) \otimes b(\mathbf{p}, 1) \right), \quad (404)$$

$$b(\mathbf{p}, N)^\dagger = \frac{1}{\sqrt{N}} \left(b(\mathbf{p}, 1)^\dagger \otimes I_m(1) \otimes \cdots \otimes I_m(1) + \cdots + I_m(1) \otimes \cdots \otimes I_m(1) \otimes b(\mathbf{p}, 1)^\dagger \right), \quad (405)$$

$$n_-(\mathbf{p}, N) = n_-(\mathbf{p}, 1) \otimes I_m(1) \otimes \cdots \otimes I_m(1) + \cdots + I_m(1) \otimes \cdots \otimes I_m(1) \otimes n_-(\mathbf{p}, 1). \quad (406)$$

Note that

$$a(\mathbf{p}, 1)^2 = |\mathbf{p}\rangle\langle\mathbf{p}| \underbrace{|\mathbf{p}\rangle\langle\mathbf{p}|}_0 \otimes \hat{a}(\mathbf{p})^2 = 0 \quad (407)$$

and similarly with products of all the other elements of HOLA provided they are taken at $\mathbf{p}' = \mathbf{p}$. In particular,

$$n_+(\mathbf{p}, 1) = \left(|\mathbf{p}\rangle\langle\mathbf{p}| \otimes \hat{a}(\mathbf{p})^\dagger \right) \left(\mathbb{I} \otimes \hat{a}(\mathbf{p}) \right) = a(\mathbf{p}, 1)^\dagger \left(\mathbb{I} \otimes \hat{a}(\mathbf{p}) \right) \neq a(\mathbf{p}, 1)^\dagger a(\mathbf{p}, 1) = 0.$$

This does not mean that the algebra is trivial or Abelian. Indeed, the 4-momentum (234)

$$\begin{aligned}
P_\mu(1) &= \int dp p_\mu |\mathbf{p}\rangle \langle \mathbf{p}| \otimes \left(\hat{a}(\mathbf{p})^\dagger \hat{a}(\mathbf{p}) + \hat{b}(\mathbf{p}) \hat{b}(\mathbf{p})^\dagger \right) \\
&= \int dp p_\mu |\mathbf{p}\rangle \langle \mathbf{p}| \otimes \left(\hat{a}(\mathbf{p})^\dagger \hat{a}(\mathbf{p}) + \hat{b}(\mathbf{p})^\dagger \hat{b}(\mathbf{p}) \right) + \int dp p_\mu |\mathbf{p}\rangle \langle \mathbf{p}| \otimes \iota_-(\mathbf{p}, \mathbf{p}), \\
&= \int dp p_\mu \left(n_+(\mathbf{p}, 1) + n_-(\mathbf{p}, 1) \right) + \int dp p_\mu I_m(\mathbf{p}, 1), \tag{408}
\end{aligned}$$

$$P_\mu(N) = P_\mu(1) \otimes I_m(1) \otimes \cdots \otimes I_m(1) + \cdots + I_m(1) \otimes \cdots \otimes I_m(1) \otimes P_\mu(1), \tag{409}$$

possesses the required properties of generator of 4-translations: For any $N \geq 1$

$$e^{iP(N)x} a(\mathbf{p}, N) e^{-iP(N)x} = a(\mathbf{p}, N) e^{-ixp}, \tag{410}$$

$$e^{iP(N)x} b(\mathbf{p}, N) e^{-iP(N)x} = b(\mathbf{p}, N) e^{-ixp}, \tag{411}$$

$$e^{iP(N)x} a(\mathbf{p}, N)^\dagger e^{-iP(N)x} = a(\mathbf{p}, N)^\dagger e^{ixp}, \tag{412}$$

$$e^{iP(N)x} b(\mathbf{p}, N)^\dagger e^{-iP(N)x} = b(\mathbf{p}, N)^\dagger e^{ixp}, \tag{413}$$

$$e^{iP(N)x} \psi(0, N) e^{-iP(N)x} = \psi(x, N), \tag{414}$$

$$e^{-iP(N)y} \psi(x, N) e^{iP(N)y} = \psi(x - y, N). \tag{415}$$

Here $\psi(x, N)$ denotes $\psi(x)$ but taken in the reducible $N \geq 1$ representation of HOLA. The vacuum term

$$P_\mu(N)_{\text{vacuum}} = N \int dp p_\mu I_m(\mathbf{p}, N) \tag{416}$$

is well defined and commutes with all elements of HOLA. In analogy to the standard Fock-space formalism we can work with

$$: P_\mu(N) : = P_\mu(N) - P_\mu(N)_{\text{vacuum}} = \int dp p_\mu \left(n_+(\mathbf{p}, N) + n_-(\mathbf{p}, N) \right) \tag{417}$$

but now, as opposed to the standard approach, the subtraction is well defined.

Remark: We know that one is not allowed to divide by 0. Why? Because then arithmetics would become ambiguous. Think of $1 \cdot 0 = 2 \cdot 0$. If division by 0 would be acceptable then we would conclude that $1 = 2$. This is the actual reason why $1/0$ is not allowed. The same situation occurs with $1 + \infty = 2 + \infty$. If we could subtract infinities then $1 = 2$. So subtraction of infinities is as forbidden in arithmetics as division by 0.▲

Remark: Due to the property of M-shaped deltas,

$$\int dp p_\mu \left(a(\mathbf{p}, 1)^\dagger a(\mathbf{p}, 1) + b(\mathbf{p}, 1) b(\mathbf{p}, 1)^\dagger \right) = 0$$

so this would not be a correct definition of free-field 4-momentum. For higher N

$$\begin{aligned} & \int dp p_\mu \left(a(\mathbf{p}, N)^\dagger a(\mathbf{p}, N) + b(\mathbf{p}, N) b(\mathbf{p}, N)^\dagger \right) \\ &= \int dp p_\mu \left(a(\mathbf{p}, N)^\dagger a(\mathbf{p}, N) + b(\mathbf{p}, N)^\dagger b(\mathbf{p}, N) \right) \neq 0 \end{aligned}$$

but for reasons mentioned earlier I do not think this is the correct free-field Hamiltonian.

Although, who knows...▲

B. Reducible $N \geq 1$ representation of HOLA — an indefinite-frequency harmonic oscillator representation of $\phi(x)$

The constant in the Noether current is $C = 1/2$. Take $\hat{c}(\mathbf{k}) = a$. For $N = 1$

$$c(\mathbf{k}, 1) = |\mathbf{k}\rangle\langle\mathbf{k}| \otimes \hat{a}, \quad (418)$$

$$c(\mathbf{k}, 1)^\dagger = |\mathbf{k}\rangle\langle\mathbf{k}| \otimes \hat{a}^\dagger, \quad (419)$$

$$I_0(\mathbf{k}, 1) = |\mathbf{k}\rangle\langle\mathbf{k}| \otimes 1, \quad (420)$$

$$n_0(\mathbf{k}, 1) = |\mathbf{k}\rangle\langle\mathbf{k}| \otimes \hat{a}^\dagger \hat{a}. \quad (421)$$

For $N > 1$,

$$c(\mathbf{k}, N) = \frac{1}{\sqrt{N}} \left(c(\mathbf{k}, 1) \otimes I_0(1) \otimes \cdots \otimes I_0(1) + \cdots + I_0(1) \otimes \cdots \otimes I_0(1) \otimes c(\mathbf{k}, 1) \right), \quad (422)$$

$$c(\mathbf{k}, N)^\dagger = \frac{1}{\sqrt{N}} \left(c(\mathbf{k}, 1)^\dagger \otimes I_0(1) \otimes \cdots \otimes I_0(1) + \cdots + I_0(1) \otimes \cdots \otimes I_0(1) \otimes c(\mathbf{k}, 1)^\dagger \right), \quad (423)$$

$$n_0(\mathbf{k}, N) = n_0(\mathbf{k}, 1) \otimes I_0(1) \otimes \cdots \otimes I_0(1) + \cdots + I_0(1) \otimes \cdots \otimes I_0(1) \otimes n_0(\mathbf{k}, 1), \quad (424)$$

$$I_0(\mathbf{k}, N) = \frac{1}{N} \left(I_0(\mathbf{k}, 1) \otimes I_0(1) \otimes \cdots \otimes I_0(1) + \cdots + I_0(1) \otimes \cdots \otimes I_0(1) \otimes I_0(\mathbf{k}, 1) \right), \quad (425)$$

$$I_0(N) = \int dk I_0(\mathbf{k}, N), \quad (\text{resolution of identity}) \quad (426)$$

The indefinite-frequency-type 4-momentum

$$\begin{aligned} P_\mu(1) &= \int dk k_\mu |\mathbf{k}\rangle\langle\mathbf{k}| \otimes \hat{a}^\dagger \hat{a} + \frac{1}{2} \int dk k_\mu |\mathbf{k}\rangle\langle\mathbf{k}| \otimes 1 \\ &= \int dk k_\mu n_0(\mathbf{k}, 1) + \frac{1}{2} \int dk k_\mu I_0(\mathbf{k}, 1) \end{aligned} \quad (427)$$

acts as a generator of 4-translations

$$e^{iP(1)x}c(\mathbf{k}, 1)e^{-iP(1)x} = c(\mathbf{k}, 1)e^{-ixk}, \quad (428)$$

$$e^{iP(1)x}c(\mathbf{k}, 1)^\dagger e^{-iP(1)x} = c(\mathbf{k}, 1)^\dagger e^{ixk}, \quad (429)$$

$$e^{iP(1)x}\phi(0, 1)e^{-iP(1)x} = \phi(x, 1), \quad (430)$$

$$e^{-iP(1)y}\phi(x, 1)e^{iP(1)y} = \phi(x - y, 1). \quad (431)$$

C. Physical interpretation of field operators

Returning to canonical position operator (149)

$$Q(t, N) = \sum_{\omega} \sqrt{\frac{2\hbar}{m\omega}} \left(a_{\omega}(N)e^{-i\omega t} + a_{\omega}(N)^\dagger e^{i\omega t} \right), \quad (432)$$

and comparing it with $\phi(x, N)$ evaluated at the origin $\mathbf{x} = \mathbf{0}$,

$$\phi(t, \mathbf{0}, N) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3 2\omega(\mathbf{k})} \left(c(\mathbf{k}, N)e^{-i\omega(\mathbf{k})t} + c(\mathbf{k}, N)^\dagger e^{i\omega(\mathbf{k})t} \right), \quad (433)$$

one understands that for $N = 1$ the field at the origin is a canonical position of some indefinite-frequency operator. The field taken at an arbitrary point \mathbf{x} is obtained from the one at the origin by translation. An elementary quantum field $\phi(x, 1)$, corresponding to $N = 1$, consists of a single oscillator that exists “everywhere” in space, i.e. is in superposition of different localizations. Representations characterized by $N > 1$ have physical interpretation of a gas consisting of N indefinite-frequency noninteracting bosonic oscillators.

D. (N, N') -oscillator representation of HOLA: N massive charged and N' massless neutral oscillators

Let $\mathcal{H}_m(N)$ and $\mathcal{H}_0(N)$ be the Hilbert spaces of the above two types of $N \geq 1$ scalar field representations. The full Hilbert space appropriate for the model of interacting fields is

$$\mathcal{H}(N, N') = \mathcal{H}_m(N) \otimes \mathcal{H}_0(N') \quad (434)$$

with two in principle independent parameters N and N' . The representations are constructed by the embeddings

$$a(\mathbf{p}, N, N') = a(\mathbf{p}, N) \otimes I_0(N'), \quad (435)$$

$$c(\mathbf{k}, N, N') = I_m(N) \otimes c(\mathbf{k}, N'), \quad (436)$$

and so on.

XIV. STATES AND THEIR BOUNDARY CONDITIONS FOR (N, N') -REPRESENTATION

We have not yet needed explicit forms of analogues of vacuum, n -particle or coherent states of quantum fields. Their construction is analogous to what we have done with states of N indefinite-frequency oscillators.

Vacuum appropriate for representations (391)–(398), (418)–(421), is constructed as follows:

$$|O_m, 1\rangle = \int dp O_m(\mathbf{p}) |\mathbf{p}, 0, 0\rangle, \quad (437)$$

$$|O_0, 1\rangle = \int dk O_0(\mathbf{k}) |\mathbf{k}, 0\rangle, \quad (438)$$

$$|O, N, N'\rangle = |O_m, 1\rangle^{\otimes N} \otimes |O_0, 1\rangle^{\otimes N'}. \quad (439)$$

Vacuum states are normalizable

$$\int dp |O_m(\mathbf{p})|^2 = \int dk |O_0(\mathbf{k})|^2 = 1 \quad (440)$$

Square integrability implies, roughly speaking, vanishing of $O_m(\mathbf{p})$ and $O_0(\mathbf{k})$ in the limits $|\mathbf{p}| \rightarrow \infty$, $|\mathbf{k}| \rightarrow \infty$. This is one of those features of field quantization in (N, N') -representations that will in the end make mathematics of field quantization “sensible” (in Dirac’s sense). For massless fields, such as $O_0(\mathbf{k})$, a boundary condition has to be imposed also at $k = (k_0, \mathbf{k}) = (|\mathbf{k}|, \mathbf{k}) = 0$. For massive fields, with $p^2 = m^2 > 0$, the point $p = 0$ does not belong to the mass- m hyperboloid: $(\pm\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p}) \rightarrow (\pm m, \mathbf{0}) \neq 0$ with $|\mathbf{p}| \rightarrow 0$.

For $m = 0$ the point $\mathbf{k} = \mathbf{0}$ is Lorentz invariant. It corresponds to $k = 0$ and $k' = Lk = L0 = 0$ for any L so that $\mathbf{k}' = \mathbf{0}$ as well. The boundary condition $O_0(\mathbf{0}) = 0$ is also Lorentz invariant (in fact, it is invariant under the whole Poincaré group). Well-definiteness of field theory that involves a massless field will require a yet stronger boundary condition, namely

$$\lim_{\mathbf{k} \rightarrow \mathbf{0}} \frac{O_0(\mathbf{k})}{|\mathbf{k}|^n} = 0, \quad (441)$$

$$\int dk \frac{|O_0(\mathbf{k})|^2}{|\mathbf{k}|^n} < \infty, \quad (442)$$

for all natural n . The fact that the tip $k = 0$ of the light cone is Lorentz invariant has deep consequences for the structure of unitary representations of the Poincaré group and

geometry of the light cone. Hilbert spaces of massless $k = 0$ and $k \neq 0$ representations are completely different (the representations are induced from different little groups [44]) and there is no Lorentz transformation that could map $k = 0$ into $k' \neq 0$, even if $k'^2 = 0$. The light cone splits in a Lorentz invariant way into the tip $k_0 = 0$ and the physical cone ($k_0 \neq 0$). The requirement that the probability density $|O_0(\mathbf{k})|^2$ tends to 0 as $\mathbf{k} \rightarrow \mathbf{0}$ is thus mathematically very natural. From a physical point of view it eliminates massless fields of infinite wavelength and zero frequency.

The rest of the construction is completely analogous to what I did for N -oscillator states. (However, we have to remember that $\langle \mathbf{p} | \mathbf{p} \rangle = \langle \mathbf{k} | \mathbf{k} \rangle = 0$.) In particular, the displacement operator is defined as

$$\begin{aligned} \mathcal{D}(\alpha, \beta, \gamma, N, N') &= \exp \int d\mathbf{p} \left(\alpha(\mathbf{p}) a(\mathbf{p}, N, N')^\dagger + \beta(\mathbf{p}) b(\mathbf{p}, N, N')^\dagger - \text{H.c.} \right) \\ &\quad \times \exp \int d\mathbf{k} \left(\gamma(\mathbf{k}) c(\mathbf{k}, N, N')^\dagger - \text{H.c.} \right) \end{aligned} \quad (443)$$

$$\begin{aligned} &= \exp \int d\mathbf{p} \left(\alpha(\mathbf{p}) a(\mathbf{p}, N)^\dagger + \beta(\mathbf{p}) b(\mathbf{p}, N)^\dagger - \text{H.c.} \right) \\ &\quad \otimes \exp \int d\mathbf{k} \left(\gamma(\mathbf{k}) c(\mathbf{k}, N')^\dagger - \text{H.c.} \right) \end{aligned} \quad (444)$$

Corresponding statistics of excitations is Rényi-deformed Poissonian.

XV. RELATIVISTIC COVARIANCE

Let x be a point in Minkowski space, i.e.

$$(x^0, x^1, x^2, x^3) = (ct, x, y, z) = (x_0, -x_1, -x_2, -x_3). \quad (445)$$

In what follows we work in units with $c = 1$, so we set $x_0 = t$. The Poincaré transformation

$$x' = Lx + y, \quad (446)$$

where L is a Lorentz transformation and y a 4-vector, is equivalent to

$$x'^\alpha = L^\alpha_\beta x^\beta + y^\alpha = x^\beta L^{-1\beta}_\alpha + y^\alpha, \quad (447)$$

$$x'_\alpha = L_\alpha^\beta x_\beta + y_\alpha = x_\beta L^{-1\beta}_\alpha + y_\alpha. \quad (448)$$

Poincaré transformations are represented unitarily by operators $U(L, y, N, N')$ constructed below. To begin with, the unitary representation of 4-translations is generated by

$$P_\mu(N, N') = \int dp p_\mu \left(n_+(\mathbf{p}, N) + n_-(\mathbf{p}, N) \right) \otimes I_0(N') + I_m(N) \otimes \int dk k_\mu n_0(\mathbf{k}, N') \\ + N \int dp p_\mu I_m(\mathbf{p}, N) \otimes I_0(N') + I_m(N) \otimes \frac{N'}{2} \int dk k_\mu I_0(\mathbf{k}, N') \quad (449)$$

A. Action of the Poincaré group on field operators

A pure 4-translation $x' = x + y$ is represented by

$$U(1, y, N, N') = e^{iy^\mu P_\mu(N, N')} = e^{iyP(N, N')}. \quad (450)$$

It multiplies annihilation operators by phase factors

$$U(1, y, N, N')^\dagger a(\mathbf{p}, N, N') U(1, y, N, N') = a(\mathbf{p}, N, N') e^{ipy}, \quad (451)$$

$$U(1, y, N, N')^\dagger b(\mathbf{p}, N, N') U(1, y, N, N') = b(\mathbf{p}, N, N') e^{ipy}, \quad (452)$$

$$U(1, y, N, N')^\dagger c(\mathbf{k}, N, N') U(1, y, N, N') = c(\mathbf{k}, N, N') e^{iky}. \quad (453)$$

Recall that $py = \sqrt{\mathbf{p}^2 + m^2} y^0 - \mathbf{p} \cdot \mathbf{y}$, $ky = |\mathbf{k}| y^0 - \mathbf{k} \cdot \mathbf{y}$. Now let L be a Lorentz transformation and \mathbf{Lp} be the spacelike part of Lp . Unitary spin-0 representation of the Lorentz group is given by

$$U_m(L, 0, 1) = \int dp |\mathbf{p}\rangle \langle \mathbf{L}^{-1} \mathbf{p}| \otimes (1 \otimes 1), \quad (454)$$

$$U_0(L, 0, 1) = \int dk |\mathbf{k}\rangle \langle \mathbf{L}^{-1} \mathbf{k}| \otimes 1, \quad (455)$$

$$U(L, 0, N, N') = U_m(L, 0, 1)^{\otimes N} \otimes U_0(L, 0, 1)^{\otimes N'}. \quad (456)$$

$U(L, 0, N, N')$ changes momenta of the amplitude operators (the Doppler effect),

$$U(L, 0, N, N')^\dagger a(\mathbf{p}, N, N') U(L, 0, N, N') = a(\mathbf{L}^{-1} \mathbf{p}, N, N'), \quad (457)$$

$$U(L, 0, N, N')^\dagger b(\mathbf{p}, N, N') U(L, 0, N, N') = b(\mathbf{L}^{-1} \mathbf{p}, N, N'), \quad (458)$$

$$U(L, 0, N, N')^\dagger c(\mathbf{k}, N, N') U(L, 0, N, N') = c(\mathbf{L}^{-1} \mathbf{k}, N, N'). \quad (459)$$

Four-translations and Lorentz transformations are combined into full Poincaré transformations $(L, y) = (1, y)(L, 0)$:

$$(L, y)x = (1, y)(L, 0)x = (1, y)(Lx + 0) = 1(Lx + 0) + y = Lx + y. \quad (460)$$

Therefore

$$U(L, y, N, N') = U(1, y, N, N')U(L, 0, N, N') \quad (461)$$

is a unitary representation of a general Poincaré transformation. As an example of explicit Poincaré transformation consider the massive charged field

$$\psi(x, N, N') = \int dp \left(a(\mathbf{p}, N, N') e^{-ipx} + b(\mathbf{p}, N, N')^\dagger e^{ipx} \right). \quad (462)$$

Then

$$\begin{aligned} & U(L, y, N, N')^\dagger \psi(x, N, N') U(L, y, N, N') \\ &= \int dp \left(U(L, 0, N, N')^\dagger U(1, y, N, N')^\dagger a(\mathbf{p}, N, N') U(1, y, N, N') U(L, 0, N, N') e^{-ipx} \right. \\ &\quad \left. + U(L, 0, N, N')^\dagger U(1, y, N, N')^\dagger b(\mathbf{p}, N, N')^\dagger U(1, y, N, N') U(L, 0, N, N') e^{ipx} \right) \\ &= \int dp \left(U(L, 0, N, N')^\dagger a(\mathbf{p}, N, N') U(L, 0, N, N') e^{-ip(x-y)} \right. \\ &\quad \left. + U(L, 0, N, N')^\dagger b(\mathbf{p}, N, N')^\dagger U(L, 0, N, N') e^{ip(x-y)} \right) \\ &= \int dp \left(a(\mathbf{L}^{-1}\mathbf{p}, N, N') e^{-ip(x-y)} + b(\mathbf{L}^{-1}\mathbf{p}, N, N')^\dagger e^{ip(x-y)} \right) \\ &= \int dp \left(a(\mathbf{L}^{-1}\mathbf{p}, N, N') e^{-iLL^{-1}p(x-y)} + b(\mathbf{L}^{-1}\mathbf{p}, N, N')^\dagger e^{iLL^{-1}p(x-y)} \right) \\ &= \int dp \left(a(\mathbf{p}, N, N') e^{-iLp(x-y)} + b(\mathbf{p}, N, N')^\dagger e^{iLp(x-y)} \right) \\ &= \int dp \left(a(\mathbf{p}, N, N') e^{-ipL^{-1}(x-y)} + b(\mathbf{p}, N, N')^\dagger e^{ipL^{-1}(x-y)} \right) \\ &= \psi(L^{-1}(x-y), N, N'). \end{aligned} \quad (463)$$

I have used here the Lorentz invariance : $d(L^{-1}p) = dp$ and $Lp \cdot Lx = p \cdot x = p_\mu x^\mu$, implying

$$Lp \cdot (x - y) = Lp \cdot LL^{-1}(x - y) = p \cdot L^{-1}(x - y).$$

$L^{-1}(x - y)$ is the inverse of $Lx + y$. Indeed,

$$L^{-1}((Lx + y) - y) = x, \quad (464)$$

$$L(L^{-1}(x - y)) + y = x. \quad (465)$$

The same transformation rule holds for the massless neutral field

$$U(L, y, N, N')^\dagger \phi(x, N, N') U(L, y, N, N') = \phi(L^{-1}(x - y), N, N'). \quad (466)$$

Number operators transform as translation invariant scalar fields in momentum space

$$U(L, y, N, N')^\dagger n_\pm(\mathbf{p}, N, N') U(L, y, N, N') = n_\pm(\mathbf{L}^{-1}\mathbf{p}, N, N'), \quad (467)$$

$$U(L, y, N, N')^\dagger n_0(\mathbf{k}, N, N') U(L, y, N, N') = n_0(\mathbf{L}^{-1}\mathbf{k}, N, N'). \quad (468)$$

In consequence, the 4-momentum is a 4-vector:

$$\begin{aligned} \int dp p_\mu n_\pm(\mathbf{L}^{-1}\mathbf{p}, N, N') &= \int dp (LL^{-1}p)_\mu n_\pm(\mathbf{L}^{-1}\mathbf{p}, N, N') \\ &= \int dp (Lp)_\mu n_\pm(\mathbf{p}, N, N') \\ &= L_\mu{}^\nu \int dp p_\nu n_\pm(\mathbf{p}, N, N') \\ \int dk k_\mu n_0(\mathbf{L}^{-1}\mathbf{k}, N, N') &= \int dk (LL^{-1}k)_\mu n_0(\mathbf{L}^{-1}\mathbf{k}, N, N') \\ &= L_\mu{}^\nu \int dk k_\nu n_0(\mathbf{k}, N, N'), \end{aligned}$$

so that

$$U(L, y, N, N')^\dagger P_\mu(N, N') U(L, y, N, N') = L_\mu{}^\nu P_\nu(N, N'). \quad (469)$$

B. Action of the Poincaré group on vacuum

Vacuum states are neither exactly translation invariant (in consequence of the “zero energy” 4-momentum, which we do not neglect)

$$\begin{aligned} U(1, y, N, N') |O, N, N'\rangle &= e^{iy^\mu P_\mu(N, N')} |O, N, N'\rangle \\ &= \left(\int dp e^{iy p} O_m(\mathbf{p}) |\mathbf{p}\rangle \otimes |0, 0\rangle \right)^{\otimes N} \otimes \left(\int dk e^{iy k/2} O_0(\mathbf{k}) |\mathbf{k}\rangle \otimes |0\rangle \right)^{\otimes N'} \end{aligned}$$

nor Lorentz invariant

$$\begin{aligned}
U(L, 0, N, N')|O, N, N'\rangle &= \left(U_m(L, 0, 1)^{\otimes N} \otimes U_0(L, 0, 1)^{\otimes N'} \right) \left(|O_m, 1\rangle^{\otimes N} \otimes |O_0, 1\rangle^{\otimes N'} \right) \\
&= \left(U_m(L, 0, 1)|O_m, 1\rangle \right)^{\otimes N} \otimes \left(U_0(L, 0, 1)|O_0, 1\rangle \right)^{\otimes N'} \\
U_m(L, 0, 1)|O_m, 1\rangle &= \int dp |\mathbf{p}\rangle \langle \mathbf{L}^{-1}\mathbf{p}| \otimes (1 \otimes 1) \int dp' O_m(\mathbf{p}') |\mathbf{p}'\rangle \otimes |0, 0\rangle \\
&= \int dp \int dp' O_m(\mathbf{p}') |\mathbf{p}\rangle \langle \mathbf{L}^{-1}\mathbf{p}|\mathbf{p}'\rangle \otimes |0, 0\rangle \\
&= \int dp O_m(\mathbf{L}^{-1}\mathbf{p}) |\mathbf{p}\rangle \otimes |0, 0\rangle, \\
U_0(L, 0, 1)|O_0, 1\rangle &= \int dk |\mathbf{k}\rangle \langle \mathbf{L}^{-1}\mathbf{k}| \otimes 1 \int dk' O_0(\mathbf{k}') |\mathbf{k}'\rangle \otimes |0\rangle \\
&= \int dk \int dk' O_0(\mathbf{k}') |\mathbf{p}\rangle \langle \mathbf{L}^{-1}\mathbf{k}|\mathbf{k}'\rangle \otimes |0\rangle \\
&= \int dk O_0(\mathbf{L}^{-1}\mathbf{k}) |\mathbf{k}\rangle \otimes |0\rangle.
\end{aligned}$$

We can say that vacuum wave functions behave under Poincaré transformations as classical scalar fields on mass- m hyperboloid and light cone, respectively,

$$O_m(\mathbf{p}) \rightarrow e^{iyp} O_m(\mathbf{p}), \quad (470)$$

$$O_m(\mathbf{p}) \rightarrow O_m(\mathbf{L}^{-1}\mathbf{p}), \quad (471)$$

$$O_0(\mathbf{k}) \rightarrow O_0(\mathbf{L}^{-1}\mathbf{k}), \quad (472)$$

$$O_0(\mathbf{k}) \rightarrow e^{iyk/2} O_0(\mathbf{k}). \quad (473)$$

The “zero-energy” parts of 4-momentum can be removed by a well defined unitary transformation. Such a new vacuum will become 4-translation invariant, but one cannot do the same with Lorentz transformations. Lorentz *invariance* would require constant $O_m(\mathbf{p})$ and $O_0(\mathbf{k})$, a condition inconsistent with square-integrability

$$\int dp |O_m(\mathbf{p})|^2 = \int dk |O_0(\mathbf{k})|^2 = 1 \quad (474)$$

which we assume.

The entire subspace of vacuum states is nevertheless Poincaré invariant. It follows that the projector on the vacuum subspace

$$\Pi_0(N, N') = \left(\int dp |\mathbf{p}\rangle \langle \mathbf{p}| \otimes |0, 0\rangle \langle 0, 0| \right)^{\otimes N} \otimes \left(\int dk |\mathbf{k}\rangle \langle \mathbf{k}| \otimes |0\rangle \langle 0| \right)^{\otimes N'} \quad (475)$$

is Poincaré invariant

$$U(L, y, N, N')^\dagger \Pi_0(N, N') U(L, y, N, N') = \Pi_0(N, N'), \quad (476)$$

and commutes with $U(L, y, N, N')$,

$$\Pi_0(N, N') U(L, y, N, N') = U(L, y, N, N') \Pi_0(N, N'). \quad (477)$$

C. Vacuum in 4-position space and nontrivial boundary conditions

The fact that vacuum wave functions behave under Lorentz transformations as classical scalar fields on mass- m hyperboloid and light cone, respectively, suggests their interpretation as Fourier-space amplitudes of classical scalar fields

$$O_m(x) = \int dp \left(O_m(\mathbf{p}) e^{-ipx} + \overline{O_m(\mathbf{p})} e^{ipx} \right), \quad (478)$$

$$O_0(x) = \int dk \left(O_0(\mathbf{k}) e^{-ikx/2} + \overline{O_0(\mathbf{k})} e^{ikx/2} \right). \quad (479)$$

The fields $O_0(x)$ and $O_m(x)$ provide a means of imposing nontrivial boundary conditions associated with concrete geometry.

This is an interesting point that will be discussed later in the context of the Casimir effect and fields in cavities. At this place it is just good to know that boundary conditions should be in my approach imposed on both vacuum states and field operators.

Remark: The essential mathematical difference between $|O, N, N'\rangle$ and the vacuum $|0\rangle$ of the bosonic Fock space is related to Poincaré invariance. Uniqueness and Poincaré invariance of the vacuum state is one of those Wightman axioms of standard Fock-space based quantum field theory [45, 46] that are not satisfied by my formalism. (Note, however, that vacuum understood as the subspace of vacuum states is both unique and Poincaré invariant even in my formalism; in Wightmanian formalism this space is 1-dimensional so both types of invariance and uniqueness are then equivalent.) In the context of indefinite-frequency oscillators the state $|O, N, N'\rangle$ represents a gas of N massive and N' massless bosonic oscillator wave packets at zero temperature. As such it should be neither unique nor Poincaré invariant, similarly to states of actual Bose-Einstein condensates occurring in atomic physics (clouds of trapped cold atoms are not even translation invariant). In relativistic theories all states should be at least Poincaré *covariant*.▲

Remark: In solid-state physics one distinguishes between “particles”, such as atoms forming a given medium, and “quasi-particles” (phonons, plasmons, etc.) — quantized oscillations of the medium. The physical difference between the Bose-Einstein condensate $|O, N, N'\rangle$ and the Fock vacuum $|0\rangle$ can be better understood if one thinks of Lewenstein-You-Cooper-Burnett Fock-space formulation of atomic Bose-Einstein condensates [47]. The vacuum there corresponds to the Fockian $|0\rangle$, i.e. the state of no atoms. This is a purely formal object, a “ground state” meaning “no atoms in the trap”. A “Bose-Einstein condensate at zero temperature” is a state analogous to $|O, N, N'\rangle$. It is not necessarily the lowest energy state of the total Hamiltonian since electrons in atoms may exist in higher excited states, but rather a ground state of a part of the Hamiltonian (typically representing the center-of-mass kinetic energy). In my formalism a kind of $|0\rangle$ could correspond to $N = 0, N' = 0$ (in the first version of the formalism published in [7] I indeed considered this type of dual particle/quasi-particle Fock-type structure). n -quasi-particle states, that is those containing n excitations of the oscillators, are regarded in these notes as representing n particles. The notion of particle statistics refers only to the quasi-particle level. Numbers of quasi-particles may change in time as opposed to N and N' that remain fixed. The “medium” composed of our N and N' particles is static and does not explicitly take part in dynamical processes.▲

Remark: The so called loop diagrams I will describe in detail later correspond to “vacuum to vacuum” processes, such as annihilation of particles that were spontaneously created from vacuum. In my formalism the appropriate formulas will be obtained by sandwiching evolution operators between *projectors* on vacuum and not between vacuum *state vectors*, as would be the case in more standard approaches.▲

XVI. INTERACTION-PICTURE DYNAMICS

Let ϕ be any operator and $H = H_0 + H_1$ a Hamiltonian. Its dynamics in Heisenberg picture is given by

$$\hat{\phi}(t) = e^{iHt}\phi e^{-iHt} = U_1(t)^\dagger e^{iH_0t}\phi e^{-iH_0t}U_1(t), \quad \hat{\phi}(0) = \phi, \quad (480)$$

$$U_1(t) = e^{iH_0t}e^{-iHt}, \quad (481)$$

$$i\dot{U}_1(t) = e^{iH_0t}(-H_0 + H)e^{-iHt} = e^{iH_0t}H_1e^{-iH_0t}U_1(t). \quad (482)$$

Denoting

$$\phi(t) = e^{iH_0 t} \phi e^{-iH_0 t}, \quad (483)$$

$$H_1(t) = e^{iH_0 t} H_1 e^{-iH_0 t}, \quad (484)$$

we can write

$$\hat{\phi}(t) = U_1(t)^\dagger \phi(t) U_1(t), \quad (485)$$

$$i\dot{U}_1(t) = H_1(t) U_1(t). \quad (486)$$

Let us note that $\hat{\phi}(t)$ and $\phi(t)$ are equal at $t = 0$ — this is an example of initial condition for Heisenberg dynamics. It is often convenient to take the initial condition $\hat{\phi}(t_0) = \phi(t_0)$ at an arbitrary t_0 ,

$$\hat{\phi}(t) = e^{iH(t-t_0)} \phi(t_0) e^{-iH(t-t_0)} \quad (487)$$

$$= e^{iH(t-t_0)} e^{iH_0 t_0} \phi e^{-iH_0 t_0} e^{-iH(t-t_0)} \quad (488)$$

$$= e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \phi(t) e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \quad (489)$$

$$= V_1(t, t_0)^\dagger \phi(t) V_1(t, t_0). \quad (490)$$

The evolution operator

$$V_1(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}, \quad (491)$$

satisfies Schrödinger's equation

$$\begin{aligned} i \frac{d}{dt} V_1(t, t_0) &= -e^{iH_0(t-t_0)} H_0 e^{-iH(t-t_0)} + e^{iH_0(t-t_0)} H e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)} H_1 e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)} H_1 e^{-iH_0(t-t_0)} V_1(t, t_0) \\ &= H_1(t-t_0) V_1(t, t_0), \end{aligned} \quad (492)$$

and

$$V_1(t_0, t_0) = \mathbb{I}, \quad (493)$$

$$V_1(t, t_0) = V_1(t-t_0, 0), \quad (494)$$

but in general does not fulfill the composition property, i.e.

$$V_1(t_1, t_2) V_1(t_2, t_3) \neq V_1(t_1, t_3). \quad (495)$$

Another option is to take

$$U_1(t, t_0) = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0} \quad (496)$$

which, as opposed to $V_1(t, t_0)$, satisfies the composition property

$$U_1(t_1, t_2) U_1(t_2, t_3) = U_1(t_1, t_3). \quad (497)$$

The dynamics is now different:

$$\begin{aligned} i \frac{d}{dt} U_1(t, t_0) &= -e^{iH_0 t} H_0 e^{-iH(t-t_0)} e^{-iH_0 t_0} + e^{iH_0 t} H e^{-iH(t-t_0)} e^{-iH_0 t_0} \\ &= e^{iH_0 t} H_1 e^{-iH(t-t_0)} e^{-iH_0 t_0} \\ &= e^{iH_0 t} H_1 e^{-iH_0 t} U_1(t, t_0) \\ &= H_1(t) U_1(t, t_0). \end{aligned} \quad (498)$$

The initial condition is unchanged,

$$U_1(t_0, t_0) = \mathbb{I}, \quad (499)$$

but $U_1(t, t_0) \neq U_1(t - t_0, 0)$. Assuming $\hat{\phi}(t_0) = \phi$ we find

$$\begin{aligned} \hat{\phi}(t) &= e^{iH(t-t_0)} \phi e^{-iH(t-t_0)} \\ &= e^{-iH_0 t_0} U_1(t, t_0)^\dagger e^{iH_0 t} \phi e^{-iH_0 t} U_1(t, t_0) e^{iH_0 t_0} \\ &= e^{-iH_0 t_0} U_1(t, t_0)^\dagger \phi(t) U_1(t, t_0) e^{iH_0 t_0} \end{aligned} \quad (500)$$

Let us note that $\hat{\phi}(t_0) = \phi = \phi(0)$ whereas the case of $V_1(t, t_0)$ corresponded to $\hat{\phi}(t_0) = \phi(t_0)$.

For $t_0 = 0$ the three evolution operators, (481), (491), (496), coincide:

$$U_1(t) = U_1(t, 0) = V_1(t, 0). \quad (501)$$

Equation (498) is known as the interaction or Dirac picture. (498) and (499) are equivalent to

$$\begin{aligned} U_1(t, t_0) &= \mathbb{I} + (-i) \int_{t_0}^t d\tau_1 H_1(\tau_1) U_1(\tau_1, t_0) \\ &= \mathbb{I} + (-i) \int_{t_0}^t d\tau_1 H_1(\tau_1) \left(\mathbb{I} + (-i) \int_{t_0}^{\tau_1} d\tau_2 H_1(\tau_2) U_1(\tau_2, t_0) \right). \end{aligned}$$

By iteration

$$U_1(t, t_0) = \mathbb{I} + (-i) \int_{t_0}^t d\tau_1 H_1(\tau_1) + (-i)^2 \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \underbrace{H_1(\tau_1) H_1(\tau_2)}_{\text{loop terms originate here}} + \dots \quad (502)$$

$$= \text{Texp} \left(-i \int_{t_0}^t d\tau_1 H_1(\tau_1) \right). \quad (503)$$

$\text{Texp}(\dots)$ is the so-called time-ordered exponential.

Another popular (and equivalent) form of $\text{Texp}(\dots)$ involves the so called time-ordered products,

$$\begin{aligned} U_1(t, t_0) &= \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \dots \int_{t_0}^{\tau_{n-1}} d\tau_n H_1(\tau_1) H_1(\tau_2) \dots H_1(\tau_n) \\ &= \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^t d\tau_1 \dots \int_{t_0}^t d\tau_n \theta(\tau_1 - \tau_2) \dots \theta(\tau_{n-1} - \tau_n) H_1(\tau_1) \dots H_1(\tau_n). \end{aligned} \quad (504)$$

Now we have n integrals from t_0 to t . Taking into account all the possible permutations $(\tau_{i_1}, \dots, \tau_{i_n})$ of the n -tuple (τ_1, \dots, τ_n) and assuming that orders of integration can be interchanged (which is not always obvious, especially if t_0 or t are taken at infinity) we obtain

$$U_1(t, t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t d\tau_1 \dots \int_{t_0}^t d\tau_n T \left(H_1(\tau_1) \dots H_1(\tau_n) \right), \quad (505)$$

where the time-ordered product is defined by

$$T \left(H_1(\tau_1) \dots H_1(\tau_n) \right) = \sum_{(\tau_{i_1}, \dots, \tau_{i_n})} \theta(\tau_{i_1} - \tau_{i_2}) \dots \theta(\tau_{i_{n-1}} - \tau_{i_n}) H_1(\tau_{i_1}) \dots H_1(\tau_{i_n}). \quad (506)$$

The above forms of dynamics can be further generalized as follows. Assume $\mathcal{O} \subset \mathbb{R}^4$ is a subset of Minkowski space contained between two spacelike hyper-surfaces Σ_{τ_1} and Σ_{τ_2} . Let us also assume that there exists a 1-parameter family of hypersurfaces Σ_τ , $\mathcal{O} = \bigcup_\tau \Sigma_\tau$, such that for each $x \in \mathcal{O}$ there exists one and only one Σ_τ containing x . The family Σ_τ is termed the foliation of \mathcal{O} .

Interaction-picture Hamiltonian associated with a given foliation is defined by

$$V(\tau) = \int_{\Sigma_\tau} d\Sigma_\tau(x) \mathcal{V}(x). \quad (507)$$

$\mathcal{V}(x)$ is a Poincaré covariant interaction-Hamiltonian density,

$$U(L, y)^\dagger \mathcal{V}(x) U(L, y) = \mathcal{V}(L^{-1}(x - y)). \quad (508)$$

$U(L, y)$ is a representation of the Poincaré group — at this moment we do not have to specify exactly which representation we have in mind. $d\Sigma_\tau$ is a measure on Σ_τ satisfying

$$d^4x = d\tau d\Sigma_\tau. \quad (509)$$

Condition (509) implicitly enforces relativistic covariance of the formalism. Let $x' = Lx$ where L is a Lorentz transformation. Then

$$d^4x = dx_0 d^3x \quad (510)$$

$$= dx'_0 d^3x' \quad (511)$$

$$= d\tau \frac{d^3x}{\sqrt{1 + \mathbf{x}^2/\tau^2}} = d\tau \frac{d^3x'}{\sqrt{1 + \mathbf{x}'^2/\tau^2}} \quad (512)$$

define two hyperplane foliations $\Sigma_\tau = \{x; x_0 = \tau\}$ and $\Sigma_\tau = \{x; x'_0 = \tau\}$ of the Minkowski space, and the hyperbolic foliation $\Sigma_\tau = \{x; (x - y)^2 = \tau^2, x_0 \geq y_0\}$ of the future causal cone of some event y (the so-called Milne universe [48]). The respective measures are

$$d\Sigma_\tau(x) = d^3x, \quad (513)$$

$$d\Sigma_\tau(x') = d^3x', \quad (514)$$

$$d\Sigma_\tau(x) = \frac{d^3x}{\sqrt{1 + \mathbf{x}^2/\tau^2}} = \frac{d^3x'}{\sqrt{1 + \mathbf{x}'^2/\tau^2}} = d\Sigma_\tau(x'). \quad (515)$$

In Dirac's terminology [49] the first two foliations define *instant-form* dynamics. The hyperbolic foliation defines the *point-form* dynamics in the Milne universe. Point-form quantum optics of classical sources formulated in “my” approach to quantum field theory was described in detail in [27]. Milne universe is the exceptional case of dynamics where free-field initial condition at $\tau = 0$ is physical: At $\tau = 0$ the fields are at the boundary of the universe.

Interaction-picture evolution operator with respect to any foliation is a solution of

$$i \frac{d}{d\tau} U_1(\tau, \tau_1) = V(\tau) U_1(\tau, \tau_1), \quad (516)$$

$$U_1(\tau_1, \tau_1) = \mathbb{I}. \quad (517)$$

An important property of interaction-picture dynamics is the fact that Hamiltonians $H_1(t)$ are constructed from operators that depend on time through the free-Hamiltonian dynamics generated by H_0 .

XVII. CONCRETE CHOICE OF ν

A scalar-scalar interaction that seems formally closest to quantum electrodynamics is

$$\begin{aligned} V(\tau) &= \int_{\mathbb{R}^3} \frac{d^3x}{\sqrt{1 + \mathbf{x}^2/\tau^2}} \frac{x^\mu}{\tau} j_\mu(x, N, N') \phi(x, N, N') \\ &= \int_{\mathbb{R}^3} \frac{d^3x}{\sqrt{\tau^2 + \mathbf{x}^2}} x^\mu j_\mu(x, N, N') \phi(x, N, N') \end{aligned} \quad (518)$$

with the current given by

$$j_\mu(x, N, N') = iq\psi(x, N, N')^\dagger \partial_\mu \psi(x, N, N') - iq\partial_\mu \psi(x, N, N')^\dagger \psi(x, N, N'). \quad (519)$$

The point $x = (\sqrt{\tau^2 + \mathbf{x}^2}, \mathbf{x})$ belongs to $\Sigma_\tau = \{x; x^2 = \tau^2, x_0 \geq 0\}$, so

$$x/\tau = (\sqrt{1 + \mathbf{x}^2/\tau^2}, \mathbf{x}/\tau) \rightarrow (1, \mathbf{0}) \quad \text{with } \tau \rightarrow \infty. \quad (520)$$

As our first nontrivial exercise, leaving aside for the moment the issues of relativistic covariance but concentrating entirely on the problem of divergences in loop diagrams, we begin with a simplified version of (518),

$$V(\tau) = \int_{\mathbb{R}^3} d^3x j_0(\tau, \mathbf{x}, N, N') \phi(\tau, \mathbf{x}, N, N'). \quad (521)$$

It corresponds to the asymptotic form of (518) for $\tau \rightarrow \infty$.

One should not confuse the models with the so-called scalar electrodynamics, a gauge invariant theory describing interaction of charged spin-0 particles with electromagnetic fields. The corresponding interaction Hamiltonian is much more complicated but contains a term analogous to (518) (see example 8.6 on p. 251 in [50]).

XVIII. SIMPLIFIED HAMILTONIAN (521) IN MOMENTUM SPACE

We begin with expressing (521) in momentum space. It is this place where we have to carefully define products and integrals of operators [see the discussion accompanying (236) and (353)]. The integrals involve $dp = d^3p/[(2\pi)^3 2\sqrt{m^2 + \mathbf{p}^2}]$, $dr = d^3r/[(2\pi)^3 2\sqrt{m^2 + \mathbf{r}^2}]$,

$$dk = d^3k/[(2\pi)^3 2|\mathbf{k}|], p_0 = \sqrt{m^2 + \mathbf{p}^2}, r_0 = \sqrt{m^2 + \mathbf{r}^2}, k_0 = |\mathbf{k}|.$$

$$\begin{aligned}
V(\tau) &= q \int d^3x \int dp \left(a(p)e^{-ipx} + b(p)^\dagger e^{ipx} \right) i\partial_0 \int dr \left(b(r)e^{-irx} + a(r)^\dagger e^{irx} \right) \phi(x) \\
&\quad - q \int d^3x i\partial_0 \int dp \left(a(p)e^{-ipx} + b(p)^\dagger e^{ipx} \right) \int dr \left(b(r)e^{-irx} + a(r)^\dagger e^{irx} \right) \phi(x) \\
&= q \int d^3x \int dp \left(a(p)e^{-ipx} + b(p)^\dagger e^{ipx} \right) \int dr r_0 \left(b(r)e^{-irx} - a(r)^\dagger e^{irx} \right) \phi(x) \\
&\quad - q \int d^3x \int dp p_0 \left(a(p)e^{-ipx} - b(p)^\dagger e^{ipx} \right) \int dr \left(b(r)e^{-irx} + a(r)^\dagger e^{irx} \right) \phi(x) \\
&= q \int d^3x \int dp dr dk r_0 \\
&\quad \times \left(a(p)e^{-ipx} + b(p)^\dagger e^{ipx} \right) \left(b(r)e^{-irx} - a(r)^\dagger e^{irx} \right) \left(c(k)e^{-ikx} + c(k)^\dagger e^{ikx} \right) \\
&\quad - q \int d^3x \int dp dr dk p_0 \\
&\quad \times \left(a(p)e^{-ipx} - b(p)^\dagger e^{ipx} \right) \left(b(r)e^{-irx} + a(r)^\dagger e^{irx} \right) \left(c(k)e^{-ikx} + c(k)^\dagger e^{ikx} \right).
\end{aligned}$$

The above expressions should be understood in the following sense

$$\begin{aligned}
V(\tau) &= q \int d^3x \int dp dr dk r_0 a(p)b(r)c(k)e^{-i(p+r+k)x} + \dots \\
&= \lim_{n_1, n_2, n_3 \rightarrow \infty} q \int d^3x \int dp dr dk r_0 \\
&\quad \times a(p, n_1, N, N') b(r, n_2, N, N') c(k, n_3, N, N') e^{-i(p+r+k)x} + \dots
\end{aligned}$$

Here $a(p, n_1, N, N')$ is the (N, N') -oscillator extension of

$$a(p, n_1, 1) = |\mathbf{p}, n_1\rangle \langle \mathbf{p}, n_1| \otimes (\hat{a} \otimes 1) \quad (522)$$

or

$$a(p, n_1, n'_1, 1) = |\mathbf{p}, n_1\rangle \langle \mathbf{p}, n'_1| \otimes (\hat{a} \otimes 1) \quad (523)$$

(compare the discussion of spectral theorem for M-shaped Dirac deltas). Similar definitions apply to the other operators. For finite n_1, n_2, \dots we can apply (212) so that

$$\begin{aligned}
V(\tau) = & \frac{q}{2} \int dp d^3r dk a(\mathbf{p}) b(\mathbf{r}) c(\mathbf{k}) e^{-i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+\mathbf{r}^2}+|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{r}-\mathbf{p}-\mathbf{k}) \\
& - \frac{q}{2} \int d^3p dr dk a(\mathbf{p}) b(\mathbf{r}) c(\mathbf{k}) e^{-i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+\mathbf{r}^2}+|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{p}-\mathbf{r}-\mathbf{k}) \\
& + \frac{q}{2} \int dp d^3r dk a(\mathbf{p}) b(\mathbf{r}) c(\mathbf{k})^\dagger e^{-i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+\mathbf{r}^2}-|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{r}-\mathbf{p}+\mathbf{k}) \\
& - \frac{q}{2} \int d^3p dr dk a(\mathbf{p}) b(\mathbf{r}) c(\mathbf{k})^\dagger e^{-i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+\mathbf{r}^2}-|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{p}-\mathbf{r}+\mathbf{k}) \\
& - \frac{q}{2} \int dp d^3r dk a(\mathbf{p}) a(\mathbf{r})^\dagger c(\mathbf{k}) e^{-i(\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+\mathbf{r}^2}+|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{r}+\mathbf{p}+\mathbf{k}) \\
& - \frac{q}{2} \int d^3p dr dk a(\mathbf{p}) a(\mathbf{r})^\dagger c(\mathbf{k}) e^{-i(\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+\mathbf{r}^2}+|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{p}+\mathbf{r}-\mathbf{k}) \\
& - \frac{q}{2} \int dp d^3r dk a(\mathbf{p}) a(\mathbf{r})^\dagger c(\mathbf{k})^\dagger e^{-i(\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+\mathbf{r}^2}-|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{r}+\mathbf{p}-\mathbf{k}) \\
& - \frac{q}{2} \int d^3p dr dk a(\mathbf{p}) a(\mathbf{r})^\dagger c(\mathbf{k})^\dagger e^{-i(\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+\mathbf{r}^2}-|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{p}+\mathbf{r}+\mathbf{k}) \\
& + \frac{q}{2} \int dp d^3r dk b(\mathbf{p})^\dagger b(\mathbf{r}) c(\mathbf{k}) e^{-i(-\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+\mathbf{r}^2}+|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{r}+\mathbf{p}-\mathbf{k}) \\
& + \frac{q}{2} \int d^3p dr dk b(\mathbf{p})^\dagger b(\mathbf{r}) c(\mathbf{k}) e^{-i(-\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+\mathbf{r}^2}+|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{p}+\mathbf{r}+\mathbf{k}) \\
& + \frac{q}{2} \int dp d^3r dk b(\mathbf{p})^\dagger b(\mathbf{r}) c(\mathbf{k})^\dagger e^{-i(-\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+\mathbf{r}^2}-|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{r}+\mathbf{p}+\mathbf{k}) \\
& + \frac{q}{2} \int d^3p dr dk b(\mathbf{p})^\dagger b(\mathbf{r}) c(\mathbf{k})^\dagger e^{-i(-\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+\mathbf{r}^2}-|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{p}+\mathbf{r}-\mathbf{k}) \\
& - \frac{q}{2} \int dp d^3r dk b(\mathbf{p})^\dagger a(\mathbf{r})^\dagger c(\mathbf{k}) e^{-i(-\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+\mathbf{r}^2}+|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{r}-\mathbf{p}+\mathbf{k}) \\
& + \frac{q}{2} \int d^3p dr dk b(\mathbf{p})^\dagger a(\mathbf{r})^\dagger c(\mathbf{k}) e^{-i(-\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+\mathbf{r}^2}+|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{p}-\mathbf{r}+\mathbf{k}) \\
& - \frac{q}{2} \int dp d^3r dk b(\mathbf{p})^\dagger a(\mathbf{r})^\dagger c(\mathbf{k})^\dagger e^{-i(-\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+\mathbf{r}^2}-|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{r}-\mathbf{p}-\mathbf{k}) \\
& + \frac{q}{2} \int d^3p dr dk b(\mathbf{p})^\dagger a(\mathbf{r})^\dagger c(\mathbf{k})^\dagger e^{-i(-\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+\mathbf{r}^2}-|\mathbf{k}|)\tau} \delta^{(3)}(-\mathbf{p}-\mathbf{r}-\mathbf{k})
\end{aligned}$$

$$\begin{aligned}
V(\tau) = & \frac{q}{2} \int dpdk \left(a(\mathbf{p})b(-\mathbf{p}-\mathbf{k}) - a(-\mathbf{p}-\mathbf{k})b(\mathbf{p}) \right) c(\mathbf{k}) e^{-i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}+\mathbf{k})^2}+|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(-\mathbf{p}-\mathbf{k})^\dagger a(\mathbf{p})^\dagger - b(\mathbf{p})^\dagger a(-\mathbf{p}-\mathbf{k})^\dagger \right) c(\mathbf{k})^\dagger e^{i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}+\mathbf{k})^2}+|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(a(\mathbf{p})b(-\mathbf{p}+\mathbf{k}) - a(-\mathbf{p}+\mathbf{k})b(\mathbf{p}) \right) c(\mathbf{k})^\dagger e^{-i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}-\mathbf{k})^2}-|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(-\mathbf{p}+\mathbf{k})^\dagger a(\mathbf{p})^\dagger - b(\mathbf{p})^\dagger a(-\mathbf{p}+\mathbf{k})^\dagger \right) c(\mathbf{k}) e^{i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}-\mathbf{k})^2}-|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(\mathbf{p}+\mathbf{k})^\dagger b(\mathbf{p}) - \underline{a(\mathbf{p})a(\mathbf{p}+\mathbf{k})^\dagger} \right) c(\mathbf{k}) e^{-i(\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+(\mathbf{p}+\mathbf{k})^2}+|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(\mathbf{p})^\dagger b(\mathbf{p}+\mathbf{k}) - \underline{a(\mathbf{p}+\mathbf{k})a(\mathbf{p})^\dagger} \right) c(\mathbf{k})^\dagger e^{i(\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+(\mathbf{p}+\mathbf{k})^2}+|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(\mathbf{p})^\dagger b(\mathbf{p}-\mathbf{k}) - \underline{a(\mathbf{p}-\mathbf{k})a(\mathbf{p})^\dagger} \right) c(\mathbf{k}) e^{-i(-\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}-\mathbf{k})^2}+|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(\mathbf{p}-\mathbf{k})^\dagger b(\mathbf{p}) - \underline{a(\mathbf{p})a(\mathbf{p}-\mathbf{k})^\dagger} \right) c(\mathbf{k})^\dagger e^{i(-\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}-\mathbf{k})^2}+|\mathbf{k}|)\tau}.
\end{aligned}$$

I have underlined the four terms that are not normally ordered. Defining the normally ordered part of $V(\tau)$ by

$$\begin{aligned}
: V(\tau) : = & \frac{q}{2} \int dpdk \left(a(\mathbf{p})b(-\mathbf{p}-\mathbf{k}) - a(-\mathbf{p}-\mathbf{k})b(\mathbf{p}) \right) c(\mathbf{k}) e^{-i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}+\mathbf{k})^2}+|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(-\mathbf{p}-\mathbf{k})^\dagger a(\mathbf{p})^\dagger - b(\mathbf{p})^\dagger a(-\mathbf{p}-\mathbf{k})^\dagger \right) c(\mathbf{k})^\dagger e^{i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}+\mathbf{k})^2}+|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(a(\mathbf{p})b(-\mathbf{p}+\mathbf{k}) - a(-\mathbf{p}+\mathbf{k})b(\mathbf{p}) \right) c(\mathbf{k})^\dagger e^{-i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}-\mathbf{k})^2}-|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(-\mathbf{p}+\mathbf{k})^\dagger a(\mathbf{p})^\dagger - b(\mathbf{p})^\dagger a(-\mathbf{p}+\mathbf{k})^\dagger \right) c(\mathbf{k}) e^{i(\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}-\mathbf{k})^2}-|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(\mathbf{p}+\mathbf{k})^\dagger b(\mathbf{p}) - \underline{a(\mathbf{p}+\mathbf{k})^\dagger a(\mathbf{p})} \right) c(\mathbf{k}) e^{-i(\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+(\mathbf{p}+\mathbf{k})^2}+|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(\mathbf{p})^\dagger b(\mathbf{p}+\mathbf{k}) - \underline{a(\mathbf{p})^\dagger a(\mathbf{p}+\mathbf{k})} \right) c(\mathbf{k})^\dagger e^{i(\sqrt{m^2+\mathbf{p}^2}-\sqrt{m^2+(\mathbf{p}+\mathbf{k})^2}+|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(\mathbf{p})^\dagger b(\mathbf{p}-\mathbf{k}) - \underline{a(\mathbf{p})^\dagger a(\mathbf{p}-\mathbf{k})} \right) c(\mathbf{k}) e^{-i(-\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}-\mathbf{k})^2}+|\mathbf{k}|)\tau} \\
& + \frac{q}{2} \int dpdk \left(b(\mathbf{p}-\mathbf{k})^\dagger b(\mathbf{p}) - \underline{a(\mathbf{p}-\mathbf{k})^\dagger a(\mathbf{p})} \right) c(\mathbf{k})^\dagger e^{i(-\sqrt{m^2+\mathbf{p}^2}+\sqrt{m^2+(\mathbf{p}-\mathbf{k})^2}+|\mathbf{k}|)\tau},
\end{aligned}$$

where the underlined terms are reordered so that annihilation operators are to the right of creation operators, one finds

$$\begin{aligned}
V(\tau) &= : V(\tau) : \\
&\quad - \frac{q}{2} \int dp dk I_m(\mathbf{p}) 2(2\pi)^3 \sqrt{m^2 + \mathbf{p}^2} \delta^{*(3)}(\mathbf{k}) c(\mathbf{k}) e^{-i(\sqrt{m^2 + \mathbf{p}^2} - \sqrt{m^2 + (\mathbf{p} + \mathbf{k})^2} + |\mathbf{k}|)\tau} \\
&\quad - \frac{q}{2} \int dp dk I_m(\mathbf{p}) 2(2\pi)^3 \sqrt{m^2 + \mathbf{p}^2} \delta^{*(3)}(\mathbf{k}) c(\mathbf{k})^\dagger e^{i(\sqrt{m^2 + \mathbf{p}^2} - \sqrt{m^2 + (\mathbf{p} + \mathbf{k})^2} + |\mathbf{k}|)\tau} \\
&\quad - \frac{q}{2} \int dp dk I_m(\mathbf{p}) 2(2\pi)^3 \sqrt{m^2 + \mathbf{p}^2} \delta^{*(3)}(\mathbf{k}) c(\mathbf{k}) e^{-i(-\sqrt{m^2 + \mathbf{p}^2} + \sqrt{m^2 + (\mathbf{p} - \mathbf{k})^2} + |\mathbf{k}|)\tau} \\
&\quad - \frac{q}{2} \int dp dk I_m(\mathbf{p}) 2(2\pi)^3 \sqrt{m^2 + \mathbf{p}^2} \delta^{*(3)}(\mathbf{k}) c(\mathbf{k})^\dagger e^{i(-\sqrt{m^2 + \mathbf{p}^2} + \sqrt{m^2 + (\mathbf{p} - \mathbf{k})^2} + |\mathbf{k}|)\tau} \\
&= : V(\tau) : \\
&\quad - q \int \frac{d^3 p}{(2\pi)^3 2\sqrt{m^2 + \mathbf{p}^2}} I_m(\mathbf{p}) (2\pi)^3 2\sqrt{m^2 + \mathbf{p}^2} \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \left(c(\mathbf{k}) + c(\mathbf{k})^\dagger \right)
\end{aligned} \tag{524}$$

Let us note that in the standard approach, where $I_m(\mathbf{p}) = 1$, the vacuum term

$$V(\tau) - : V(\tau) : \tag{525}$$

is badly divergent due to the first integral

$$\int \frac{d^3 p}{(2\pi)^3 2\sqrt{m^2 + \mathbf{p}^2}} I_m(\mathbf{p}) (2\pi)^3 2\sqrt{m^2 + \mathbf{p}^2} = \int_{\mathbb{R}^3} d^3 p. \tag{526}$$

This is precisely an example of “ultraviolet catastrophe” since the integral is divergent due to the behavior of its integrand (equal to 1) for large momenta.

The other term

$$\int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{(3)}(\mathbf{k}) \left(c(\mathbf{k}) + c(\mathbf{k})^\dagger \right) = \lim_{\mathbf{k} \rightarrow 0} \frac{c(\mathbf{k}) + c(\mathbf{k})^\dagger}{(2\pi)^3 2|\mathbf{k}|} \tag{527}$$

would be in standard approaches an example of “infrared catastrophe”. Eq. (527) shows that $c(\mathbf{k}) + c(\mathbf{k})^\dagger$ must tend to zero at least as fast as $|\mathbf{k}|$ but in the standard formalism it is completely unclear how to achieve it in a mathematically precise way. The “infrared catastrophe” is typical of massless quantum fields — $|\mathbf{k}|$ in denominators would be replaced for $m > 0$ by $\sqrt{m^2 + \mathbf{k}^2}$.

The most typical way of dealing with the divergences occurring in $V(\tau) - : V(\tau) :$ is just to ignore them! We ignore them not because they are small, but “because we do not want them”. One replaces $V(\tau)$ by $: V(\tau) :$ and starts with

$$i \frac{d}{d\tau} U_1(\tau, \tau_1) = : V(\tau) : U_1(\tau, \tau_1). \tag{528}$$

It is not hard to guess that this type of “solution” will sooner or later lead to another “catastrophe” that will have to be remedied in one way or another.

The elegance typical of quantum mechanics has been completely lost. What one does is not “sensible mathematics”. This is not even a physical theory but a “collection of working rules” — at least in Dirac’s opinion.

Let us now see what happens in (N, N') representations of HOLA. First of all, by (402)

$$\begin{aligned} I_m(\mathbf{p}, N, N') &= I_m(\mathbf{p}, N) \otimes I_0(N') \\ &= \frac{1}{N} \left((|\mathbf{p}\rangle\langle\mathbf{p}| \otimes (1 \otimes 1)) \otimes I_m(1) \otimes \cdots \otimes I_m(1) + \dots \right) \otimes I_0(N'), \end{aligned}$$

The expression

$$\int \frac{d^3p}{(2\pi)^3 2\sqrt{m^2 + \mathbf{p}^2}} I_m(\mathbf{p}, N) (2\pi)^3 2\sqrt{m^2 + \mathbf{p}^2} \quad (529)$$

is $1/N$ times a sum of N operators of the form

$$\int \frac{d^3p}{(2\pi)^3 2\sqrt{m^2 + \mathbf{p}^2}} I_m(\mathbf{p}, 1) (2\pi)^3 2\sqrt{m^2 + \mathbf{p}^2} = (2\pi)^3 2 \int dp |\mathbf{p}\rangle\langle\mathbf{p}| \sqrt{m^2 + \mathbf{p}^2} \otimes (1 \otimes 1) \quad (530)$$

each of them acting in the Hilbert space of a single indefinite-frequency oscillator. (530) is the spectral representation of a well behaved operator

$$(2\pi)^3 2 \sqrt{m^2 + \hat{\mathbf{p}}^2} \otimes (1 \otimes 1), \quad (531)$$

where

$$\hat{\mathbf{p}} = \int dp |\mathbf{p}\rangle\langle\mathbf{p}| \mathbf{p}. \quad (532)$$

Recall that $\langle\mathbf{p}|\mathbf{p}\rangle = 0$, a fact that will become important later.

The example shows “regularization by quantization” in action: There is no “ultraviolet catastrophe” because the ordinary integral

$$\int dp (\dots) \quad (533)$$

has been replaced by the *spectral integral*

$$\int dp |\mathbf{p}\rangle\langle\mathbf{p}| (\dots) \quad (534)$$

Operators (531) and (532) are the analogues of, respectively,

$$\Omega = \sum_{\omega} \omega |\omega\rangle \langle \omega| \otimes I, \quad (\text{compare (97)})$$

and $\hat{\omega} = \sum_{\omega} \omega |\omega\rangle \langle \omega|$ we have started with in our discussion of indefinite-frequency oscillators.

Now let us discuss the issue of infrared divergence of

$$\begin{aligned} & \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \left(c(\mathbf{k}, N, N') + c(\mathbf{k}, N, N')^\dagger \right) \\ &= I_m(N) \otimes \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \left(c(\mathbf{k}, N') + c(\mathbf{k}, N')^\dagger \right). \end{aligned} \quad (535)$$

This is essentially a sum of N' single-oscillator operators

$$\int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \left(c(\mathbf{k}, 1) + c(\mathbf{k}, 1)^\dagger \right) = \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) |\mathbf{k}\rangle \langle \mathbf{k}| \otimes (a + a^\dagger) \quad (536)$$

multiplied by $1/\sqrt{N'}$. Now let us act with it on a single-oscillator ($N' = 1$) vacuum state

$$\begin{aligned} \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) |\mathbf{k}\rangle \langle \mathbf{k}| \otimes (a + a^\dagger) |O_0, 1\rangle &= \int dk \delta^{*(3)}(\mathbf{k}) |\mathbf{k}\rangle \langle \mathbf{k}| \otimes a^\dagger \int dk' O_0(\mathbf{k}') |\mathbf{k}', 0\rangle \\ &= \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) O_0(\mathbf{k}) |\mathbf{k}, 1\rangle \\ &= \lim_{\mathbf{k} \rightarrow \mathbf{0}} \frac{1}{(2\pi)^3 2|\mathbf{k}|} O_0(\mathbf{k}) |\mathbf{k}, 1\rangle = 0. \end{aligned}$$

The latter follows from the Poincaré invariant boundary condition (441)

$$\lim_{\mathbf{k} \rightarrow \mathbf{0}} \frac{O_0(\mathbf{k})}{|\mathbf{k}|^n} = 0.$$

Now check the action of (536) on a more general “one-photon” one-oscillator state

$$|\psi\rangle = \int dk' \psi(\mathbf{k}') c(\mathbf{k}', 1)^\dagger |O_0, 1\rangle, \quad (537)$$

$$\begin{aligned}
& \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \left(c(\mathbf{k}, 1) + c(\mathbf{k}, 1)^\dagger \right) |\psi\rangle \\
&= \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) c(\mathbf{k}, 1) \int dk' \psi(\mathbf{k}') c(\mathbf{k}', 1)^\dagger |O_0, 1\rangle \\
&\quad + \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) c(\mathbf{k}, 1)^\dagger \int dk' \psi(\mathbf{k}') c(\mathbf{k}', 1) |O_0, 1\rangle \\
&= \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \int dk' \psi(\mathbf{k}') \delta_0(\mathbf{k}, \mathbf{k}') I_0(\mathbf{k}, 1) \int dk'' O_0(\mathbf{k}'') |\mathbf{k}''\rangle \otimes |0\rangle \\
&\quad + \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) c(\mathbf{k}, 1)^\dagger \int dk' \psi(\mathbf{k}') c(\mathbf{k}', 1)^\dagger \int dk'' O_0(\mathbf{k}'') |\mathbf{k}''\rangle \otimes |0\rangle \\
&= \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \int dk' \psi(\mathbf{k}') \delta_0(\mathbf{k}, \mathbf{k}') \int dk'' O_0(\mathbf{k}'') |\mathbf{k}\rangle \langle \mathbf{k} | \mathbf{k}'' \rangle \otimes |0\rangle \\
&\quad + \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \int dk' \psi(\mathbf{k}') \int dk'' O_0(\mathbf{k}'') |\mathbf{k}\rangle \langle \mathbf{k} | \mathbf{k}' \rangle \langle \mathbf{k}' | \mathbf{k}'' \rangle \otimes |2\rangle \\
&= \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \int dk' \psi(\mathbf{k}') \delta_0(\mathbf{k}, \mathbf{k}') \int dk'' O_0(\mathbf{k}'') \delta_0(\mathbf{k}, \mathbf{k}'') |\mathbf{k}\rangle \otimes |0\rangle \\
&\quad + \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \int dk' \psi(\mathbf{k}') \int dk'' O_0(\mathbf{k}'') \delta_0(\mathbf{k}, \mathbf{k}') \delta_0(\mathbf{k}', \mathbf{k}'') |\mathbf{k}\rangle \otimes |2\rangle \\
&= \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \psi(\mathbf{k}) O_0(\mathbf{k}) |\mathbf{k}\rangle \otimes |0\rangle + \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \delta^{*(3)}(\mathbf{k}) \psi(\mathbf{k}) O_0(\mathbf{k}) |\mathbf{k}\rangle \otimes |2\rangle \\
&= \lim_{\mathbf{k} \rightarrow \mathbf{0}} \frac{1}{(2\pi)^3 2|\mathbf{k}|} \psi(\mathbf{k}) O_0(\mathbf{k}) |\mathbf{k}\rangle \otimes (|0\rangle + |2\rangle).
\end{aligned}$$

The term again vanishes unless $\psi(\mathbf{k})$ blows up at the origin faster than any power of $1/|\mathbf{k}|$. The argument can be generalized by induction to states generated from vacuum by any number of creation oscillators.

All of this is possible since in the formula

$$O_0(\mathbf{k}) = \langle O_0, N' | I_0(\mathbf{k}, N') | O_0, N' \rangle \quad (538)$$

the operator $I_0(\mathbf{k}, N')$ is not proportional to the identity. In representations occurring in the standard formalisms one would find similar expressions as we have obtained (since algebraic manipulations with operators would be identical), but with

$$O_0(\mathbf{k}) = \langle O | I(\mathbf{k}) | O \rangle = \langle O | I | O \rangle = 1 \quad (539)$$

and then infrared divergences would be inevitable.

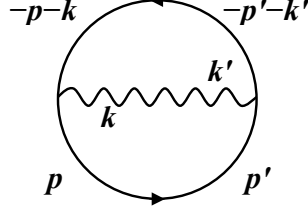


FIG. 6: The simplest loop vacuum-to-vacuum Feynman diagram represents a process where creation operators $a(\mathbf{p})^\dagger$, $b(-\mathbf{p} - \mathbf{k})^\dagger$, $c(\mathbf{k})^\dagger$ act on vacuum and the resulting state is then acted upon by the annihilation operators $a(\mathbf{p}')$, $b(-\mathbf{p}' - \mathbf{k}')$, $c(\mathbf{k}')$. So one starts from vacuum and ends up with vacuum.

XIX. VACUUM-TO-VACUUM LOOP DIAGRAM IN 2ND ORDER PERTURBATION THEORY

Vacuum-to-vacuum loop diagram is the 2nd order perturbative correction to $\langle O, N, N' | U_1(\tau_2, \tau_1) | O, N, N' \rangle$, where $|O, N, N'\rangle$ is a vacuum space. What is very important we want to have a result that does not depend on concrete explicit forms of vacuum wave functions $O_m(\mathbf{p})$ and $O_0(\mathbf{k})$ but only on their general properties such as boundary conditions.

So let $\Pi_0 = \Pi_0(N, N')$ be the projector on the vacuum subspace. It is characterized by

$$a(\mathbf{p})\Pi_0 = b(\mathbf{p})\Pi_0 = c(\mathbf{k})\Pi_0 = \Pi_0 a(\mathbf{p})^\dagger = \Pi_0 b(\mathbf{p})^\dagger = \Pi_0 c(\mathbf{k})^\dagger = 0, \quad (540)$$

$$\int dk \delta^{*(3)}(\mathbf{k}) c(\mathbf{k})^\dagger \Pi_0 = \int dk \delta^{*(3)}(\mathbf{k}) \Pi_0 c(\mathbf{k}) = \int dk \delta^{*(3)}(\mathbf{k}) I_0(\mathbf{k}) \Pi_0 = 0. \quad (541)$$

Our goal is to compute in 2nd order perturbation theory the operator $\Pi_0 U_1(\tau_2, \tau_1) \Pi_0$. The

appropriate term is

$$\begin{aligned}
& \Pi_0 V(\tau) V(\tau') \Pi_0 \\
&= \Pi_0 : V(\tau) :: V(\tau') : \Pi_0 \\
&= \frac{q^2}{4} \int dp dk dp' dk' e^{-i(\sqrt{m^2+p^2}+\sqrt{m^2+(p+k)^2}+|k|)\tau} e^{i(\sqrt{m^2+p'^2}+\sqrt{m^2+(p'+k')^2}+|k'|)\tau'} \\
&\quad \times \Pi_0 \left(a(\mathbf{p}) b(-\mathbf{p}-\mathbf{k}) - a(-\mathbf{p}-\mathbf{k}) b(\mathbf{p}) \right) c(\mathbf{k}) \\
&\quad \times \left(b(-\mathbf{p}'-\mathbf{k}')^\dagger a(\mathbf{p}')^\dagger - b(\mathbf{p}')^\dagger a(-\mathbf{p}'-\mathbf{k}')^\dagger \right) c(\mathbf{k}')^\dagger \Pi_0 \\
&= \frac{q^2}{4} \int dp dk dp' e^{-i(\sqrt{m^2+p^2}+\sqrt{m^2+(p+k)^2}+|k|)\tau} e^{i(\sqrt{m^2+p'^2}+\sqrt{m^2+(p'+k)^2}+|k|)\tau'} \\
&\quad \times \Pi_0 \left(a(\mathbf{p}) a(\mathbf{p}')^\dagger b(-\mathbf{p}-\mathbf{k}) b(-\mathbf{p}'-\mathbf{k})^\dagger + a(-\mathbf{p}-\mathbf{k}) a(-\mathbf{p}'-\mathbf{k})^\dagger b(\mathbf{p}) b(\mathbf{p}')^\dagger \right. \\
&\quad \left. - a(\mathbf{p}) a(-\mathbf{p}'-\mathbf{k})^\dagger b(-\mathbf{p}-\mathbf{k}) b(\mathbf{p}')^\dagger - a(-\mathbf{p}-\mathbf{k}) a(\mathbf{p}')^\dagger b(\mathbf{p}) b(-\mathbf{p}'-\mathbf{k})^\dagger \right) I_0(\mathbf{k}) \Pi_0 \\
&= \frac{q^2}{4} \int dp dk dp' e^{-i(\sqrt{m^2+p^2}+\sqrt{m^2+(p+k)^2}+|k|)\tau} e^{i(\sqrt{m^2+p'^2}+\sqrt{m^2+(p'+k)^2}+|k|)\tau'} \\
&\quad \times \Pi_0 \left(I_m(\mathbf{p}) \delta_m(\mathbf{p}, \mathbf{p}') I_m(-\mathbf{p}-\mathbf{k}) \delta_m(-\mathbf{p}-\mathbf{k}, -\mathbf{p}'-\mathbf{k}) \right. \\
&\quad + I_m(-\mathbf{p}-\mathbf{k}) \delta_m(-\mathbf{p}-\mathbf{k}, -\mathbf{p}'-\mathbf{k}) I_m(\mathbf{p}) \delta_m(\mathbf{p}, \mathbf{p}') \\
&\quad - I_m(\mathbf{p}) \delta_m(\mathbf{p}, -\mathbf{p}'-\mathbf{k}) I_m(-\mathbf{p}-\mathbf{k}) \delta_m(-\mathbf{p}-\mathbf{k}, \mathbf{p}') \\
&\quad \left. - I_m(-\mathbf{p}-\mathbf{k}) \delta_m(-\mathbf{p}-\mathbf{k}, \mathbf{p}') I_m(\mathbf{p}) \delta_m(\mathbf{p}, -\mathbf{p}'-\mathbf{k}) \right) I_0(\mathbf{k}) \Pi_0 \tag{542}
\end{aligned}$$

The latter expression is ill defined in standard approaches due to occurrences of squared Dirac deltas. Additional divergences would be generated if one assumed the standard condition $I_0(\mathbf{k}) = I_m(\mathbf{p}) = 1$. This is why the loop diagram is normally an example of a “difficult infinity” (see for example an analogous calculation made in full electrodynamics in chapter 4.1 of [51]). In my formalism the calculation is rather trivial.

We have to keep in mind that $dk = \rho_0(\mathbf{k}) d^3k$, $dp = \rho_m(\mathbf{p}) d^3p$ so that internal consistency conditions analogous to (284) can be derived. The only solution is thus to work with M-shaped Dirac deltas satisfying $\delta_m(\mathbf{p}, \mathbf{p}) = 0$. The result is

$$\begin{aligned}
(542) &= \frac{q^2}{4} \int dp dk e^{-i(\sqrt{m^2+p^2}+\sqrt{m^2+(p+k)^2}+|k|)\tau} e^{i(\sqrt{m^2+p'^2}+\sqrt{m^2+(p'+k)^2}+|k|)\tau'} \\
&\quad \times \left(\delta_m(-\mathbf{p}-\mathbf{k}, -\mathbf{p}-\mathbf{k}) + \delta_m(-\mathbf{p}-\mathbf{k}, -\mathbf{p}-\mathbf{k}) - \delta_m(\mathbf{p}, \mathbf{p}) - \delta_m(\mathbf{p}, \mathbf{p}) \right) \\
&\quad \times \Pi_0 I_m(\mathbf{p}) I_m(-\mathbf{p}-\mathbf{k}) I_0(\mathbf{k}) \Pi_0 \\
&= 0. \tag{543}
\end{aligned}$$

Notice that we have not needed the explicit forms of $O_m(\mathbf{p})$ and $O_0(\mathbf{k})$. In standard quantum

field theoretic parlance the result thus “does not depend on a cutoff”.

XX. MASSLESS QUANTUM SCALAR FIELD PRODUCED BY A CLASSICAL POINTLIKE SOURCE

In order to compute radiative corrections to atomic energy levels one first has to decide what kind of potential is associated with atomic nucleus. In simplest approaches the nucleus is modeled by the classical static pointlike charge density

$$\rho(\mathbf{x}) = q\delta^{(3)}(\mathbf{x}). \quad (544)$$

Classically the field produced by the source would be a solution of an inhomogeneous (Maxwell, d’Alembert...) field equation. In the quantum context one should solve the Heisenberg equation with appropriate interaction term.

Let us employ Heisenberg and Dirac picture forms of evolution equations to the dynamics of $\hat{\phi}(t, \mathbf{x})$ for $V(t) = \int_{\mathbb{R}^3} d^3x \rho(\mathbf{x})\phi(t, \mathbf{x})$,

$$\phi(t, \mathbf{x}) = \int dk \left(c(\mathbf{k}) e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}} + c(\mathbf{k})^\dagger e^{i|\mathbf{k}|t - i\mathbf{k}\cdot\mathbf{x}} \right), \quad (545)$$

$$\begin{aligned} V(t) &= q \int_{\mathbb{R}^3} d^3x \delta^{(3)}(\mathbf{x}) \phi(t, \mathbf{x}) \\ &= q\phi(t, \mathbf{0}) \end{aligned} \quad (546)$$

$$\begin{aligned} &= q \underbrace{\int dk c(\mathbf{k}) e^{-i|\mathbf{k}|t}}_{V_-(t)} + q \underbrace{\int dk c(\mathbf{k})^\dagger e^{i|\mathbf{k}|t}}_{V_+(t)=V_-(t)^\dagger}. \end{aligned} \quad (547)$$

The example is interesting since in standard field quantization $U_1(t, t_0)$ does not exist due to ultraviolet divergences. We will need

$$\begin{aligned} [V_-(t), V_+(t')] &= q^2 \left[\int dk c(\mathbf{k}) e^{-i|\mathbf{k}|t}, \int dk' c(\mathbf{k}')^\dagger e^{i|\mathbf{k}'|t'} \right] \\ &= q^2 \int dk I_0(\mathbf{k}) e^{i|\mathbf{k}|(t'-t)}. \end{aligned} \quad (548)$$

Of particular interest is the equal-time commutator

$$[V_-(t), V_+(t)] = q^2 \int dk I_0(\mathbf{k}).$$

In reducible N -representation of HOLA

$$[V_-(t), V_+(t)] = q^2 \int dk I_0(\mathbf{k}, N) \quad (549)$$

$$= q^2 I_0(N) \quad (550)$$

is a well defined operator (549). This should be contrasted with the divergent integral that would have occurred here for $I_0(\mathbf{k}) = 1$. Since for all times

$$[V_-(t), V_-(t')] = [V_+(t), V_+(t')] = 0 \quad (551)$$

and

$$[[V_-(t), V_+(t')], V_\pm(t'')] = 0, \quad (552)$$

we can use the continuous Baker-Campbell-Hausdorff (BCH) formula (cf. Appendix H, Eq. (13) in [52]),

$$U_1(t, t_0) = \text{Texp} \left(-i \int_{t_0}^t d\tau (V_+(\tau) + V_-(\tau)) \right) \quad (553)$$

$$\begin{aligned} &= \exp \left(-i \int_{t_0}^t d\tau V_+(\tau) \right) \exp \left(-i \int_{t_0}^t d\tau V_-(\tau) \right) \\ &\quad \times \exp \left(\int_{t_0}^t d\tau \int_{t_0}^\tau d\tau' [V_+(\tau'), V_-(\tau)] \right). \end{aligned} \quad (554)$$

Another useful form of continuous BCH identity is based on the ordinary BCH identity $e^X e^Y = e^{[X,Y]/2} e^{X+Y}$, which can be applied to the right-hand-side of (554),

$$\begin{aligned} U_1(t, t_0) &= \exp \left(-i \int_{t_0}^t d\tau V(\tau) \right) \exp \left(-\frac{1}{2} \int_{t_0}^t d\tau \int_{t_0}^\tau d\tau' [V_+(\tau'), V_-(\tau)] \right) \\ &\quad \times \exp \left(\int_{t_0}^t d\tau \int_{t_0}^\tau d\tau' \theta(\tau - \tau') [V_+(\tau'), V_-(\tau)] \right) \\ &= \exp \left(-i \int_{t_0}^t d\tau V(\tau) \right) \exp \left(\frac{1}{2} \int_{t_0}^t d\tau \int_{t_0}^\tau d\tau' \text{sgn}(\tau - \tau') [V_+(\tau'), V_-(\tau)] \right). \end{aligned} \quad (555)$$

(556)

$\text{sgn}(x)$ and $\theta(x)$ are, respectively, the sign-of- x and step functions. For $\tau = \tau'$ the commutators in exponents are finite in “my” representation of HOLA, so exact values at zero of sgn and θ are for the moment irrelevant. Both exponents at the right side of (556) are unitary. The second of them commutes with all the elements of HOLA, so this is an operator that in practice behaves as numerical phase factor.

Continuous BCH formula can be verified in a straightforward manner. First of all, at $t = t_0$ the left and right sides of (554) equal \mathbb{I} . It remains to check if they satisfy the same

differential equation,

$$\begin{aligned}
i \frac{dU_1(t, t_0)}{dt} &= V_+(t) \exp \left(-i \int_{t_0}^t d\tau V_+(\tau) \right) \exp \left(-i \int_{t_0}^t d\tau V_-(\tau) \right) \\
&\quad \times \exp \left(\int_{t_0}^t d\tau \int_{t_0}^\tau d\tau' [V_+(\tau'), V_-(\tau)] \right) \\
&\quad + \exp \left(-i \int_{t_0}^t d\tau V_+(\tau) \right) V_-(t) \exp \left(-i \int_{t_0}^t d\tau V_-(\tau) \right) \\
&\quad \times \exp \left(\int_{t_0}^t d\tau \int_{t_0}^\tau d\tau' [V_+(\tau'), V_-(\tau)] \right) \\
&\quad + i \int_{t_0}^t d\tau' [V_+(\tau'), V_-(t)] \exp \left(-i \int_{t_0}^t d\tau V_+(\tau) \right) \exp \left(-i \int_{t_0}^t d\tau V_-(\tau) \right) \\
&\quad \times \exp \left(\int_{t_0}^t d\tau \int_{t_0}^\tau d\tau' [V_+(\tau'), V_-(\tau)] \right) \\
&= \left(V_+(t) + e^{-i \int_{t_0}^t d\tau V_+(\tau)} V_-(t) e^{i \int_{t_0}^t d\tau V_+(\tau)} + i \int_{t_0}^t d\tau' [V_+(\tau'), V_-(t)] \right) U_1(t, t_0)
\end{aligned}$$

Employing (53) and (552) we rewrite the second term

$$e^{-i \int_{t_0}^t d\tau V_+(\tau)} V_-(t) e^{i \int_{t_0}^t d\tau V_+(\tau)} = V_-(t) + \left[-i \int_{t_0}^t d\tau V_+(\tau), V_-(t) \right],$$

which ends the proof:

$$i\dot{U}_1(t, t_0) = (V_+(t) + V_-(t))U_1(t, t_0). \quad (557)$$

We can finally compute

$$\begin{aligned}
\hat{\phi}(t, \mathbf{x}) &= \exp \left(i \int_{t_0}^t d\tau V(\tau) \right) \phi(t, \mathbf{x}) \exp \left(-i \int_{t_0}^t d\tau V(\tau) \right) \\
&= \exp \left(iq \int_{t_0}^t d\tau \phi(\tau, \mathbf{0}) \right) \phi(t, \mathbf{x}) \exp \left(-iq \int_{t_0}^t d\tau \phi(\tau, \mathbf{0}) \right) \\
&= \phi(t, \mathbf{x}) + iq \int_{t_0}^t d\tau [\phi(\tau, \mathbf{0}), \phi(t, \mathbf{x})] \\
&= \phi(t, \mathbf{x}) \\
&\quad + iq \int_{t_0}^t d\tau \int d\mathbf{k} d\mathbf{k}' \\
&\quad \times [c(\mathbf{k}) e^{-i|\mathbf{k}|\tau} + c(\mathbf{k})^\dagger e^{i|\mathbf{k}|\tau}, c(\mathbf{k}') e^{-i|\mathbf{k}'|t+i\mathbf{k}'\cdot\mathbf{x}} + c(\mathbf{k}')^\dagger e^{i|\mathbf{k}'|t-i\mathbf{k}'\cdot\mathbf{x}}] \\
&= \phi(t, \mathbf{x}) + iq \int d\mathbf{k} I_0(\mathbf{k}) \left(e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}} \int_{t_0}^t d\tau e^{-i|\mathbf{k}|\tau} - e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}} \int_{t_0}^t d\tau e^{i|\mathbf{k}|\tau} \right) \\
&= \phi(t, \mathbf{x}) \\
&\quad - q \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|^2} I_0(\mathbf{k}) \left(e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}} (e^{-i|\mathbf{k}|t} - e^{-i|\mathbf{k}|t_0}) + e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}} (e^{i|\mathbf{k}|t} - e^{i|\mathbf{k}|t_0}) \right) \\
&= \phi(t, \mathbf{x}) - q \int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|^2} I_0(\mathbf{k}) \left(e^{-i\mathbf{k}\cdot\mathbf{x}} + e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{k}x} e^{-i|\mathbf{k}|t_0} - e^{-i\mathbf{k}x} e^{i|\mathbf{k}|t_0} \right).
\end{aligned}$$

$\hat{\phi}(t, \mathbf{x})$ splits into free-field and source parts. The source term can be further split into the “generalized Coulomb field”

$$\phi_{\text{gC}}(\mathbf{x}) = -q \int \frac{d^3k}{(2\pi)^3 |\mathbf{k}|^2} I_0(\mathbf{k}) \cos \mathbf{k} \cdot \mathbf{x} \quad (558)$$

and the “compensating field”

$$\phi_{\text{c}}(t - t_0, \mathbf{x}) = q \int \frac{d^3k}{(2\pi)^3 |\mathbf{k}|^2} I_0(\mathbf{k}) \cos(kx - |\mathbf{k}|t_0). \quad (559)$$

Compensating and Coulomb fields cancel each another at $t = t_0$. Let us discuss the two terms separately.

A. Compensating field

In order to better understand the role of the compensating field let us consider an arbitrary coherent state

$$|\alpha, N\rangle = \mathcal{D}_0(\alpha, N)|O_0, N\rangle, \quad (560)$$

$$\mathcal{D}_0(\alpha, N) = \exp \int dk \left(\alpha(\mathbf{k}) c(\mathbf{k}, N)^\dagger - \overline{\alpha(\mathbf{k})} c(\mathbf{k}, N) \right). \quad (561)$$

Since

$$\mathcal{D}_0(\alpha, N)^\dagger I_0(\mathbf{k}, N) \mathcal{D}_0(\alpha, N) = I_0(\mathbf{k}, N) \quad (562)$$

one finds

$$\begin{aligned} \langle \alpha, N | \phi_{\text{c}}(t - t_0, \mathbf{x}, N) | \alpha, N \rangle &= q \int \frac{d^3k}{(2\pi)^3 |\mathbf{k}|^2} \langle O_0, N | I_0(\mathbf{k}, N) | O_0, N \rangle \cos(kx - |\mathbf{k}|t_0) \\ &= q \int \frac{d^3k}{(2\pi)^3 |\mathbf{k}|^2} |O_0(\mathbf{k})|^2 \cos(kx - |\mathbf{k}|t_0). \end{aligned}$$

The same result will be obtained if we replace the displacement operator by any unitary operator constructed from elements of HOLA. Normalization and boundary conditions imposed on vacuum imply

$$\int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} |O_0(\mathbf{k})|^2 = 1, \quad (563)$$

$$\lim_{\mathbf{k} \rightarrow \mathbf{0}} \frac{|O_0(\mathbf{k})|^2}{|\mathbf{k}|^2} = 0. \quad (564)$$

If additionally

$$\int \frac{d^3 k}{(2\pi)^3 2|\mathbf{k}|} \frac{|O_0(\mathbf{k})|^2}{|\mathbf{k}|} < \infty \quad (565)$$

then Riemann-Lebesgue lemma (Chapter 2.5, Theorem 2.2 in [53]) implies

$$\begin{aligned} \lim_{t_0 \rightarrow \pm\infty} \int \frac{d^3 k}{(2\pi)^3 |\mathbf{k}|^2} |O_0(\mathbf{k})|^2 \cos(kx) \cos(|\mathbf{k}|t_0) &= 0 \\ \lim_{t_0 \rightarrow \pm\infty} \int \frac{d^3 k}{(2\pi)^3 |\mathbf{k}|^2} |O_0(\mathbf{k})|^2 \sin(kx) \sin(|\mathbf{k}|t_0) &= 0. \end{aligned}$$

Hence

$$\lim_{t_0 \rightarrow \pm\infty} \langle \psi, N | \phi_c(t - t_0, \mathbf{x}, N) | \psi, N \rangle = 0 \quad (566)$$

for all states $|\psi, N\rangle$ generated from vacuum by unitary transformations constructed from the N -representation of HOLA. In this weak sense one can write

$$\lim_{t_0 \rightarrow \pm\infty} \phi_c(t - t_0, \mathbf{x}, N) = 0. \quad (567)$$

Let us note that asymptotic vanishing of compensating fields leads to the additional condition (565) that should be satisfied by vacuum wave functions.

For comparison let us check what happens in the usual representation, $I_0(\mathbf{k}) = 1$,

$$\begin{aligned} \phi_c(t - t_0, \mathbf{x}) &= q \int \frac{d^3 k}{(2\pi)^3 |\mathbf{k}|^2} \cos(kx - |\mathbf{k}|t_0) \\ &= q \int \frac{d^3 k}{(2\pi)^3 |\mathbf{k}|^2} \cos(|\mathbf{k}|(t - t_0) - \mathbf{k} \cdot \mathbf{x}) \\ &= \frac{q}{(2\pi)^3} \int_0^\infty d\kappa \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \cos(\kappa(t - t_0) - |\mathbf{x}| \kappa \cos \theta) \\ &= -\frac{q}{(2\pi)^2} \int_0^\infty d\kappa \int_1^{-1} du \cos(\kappa(t - t_0) - |\mathbf{x}| \kappa u) \\ &= \frac{q}{(2\pi)^2} \int_0^\infty d\kappa \frac{1}{|\mathbf{x}| \kappa} \sin(\kappa(t - t_0) - |\mathbf{x}| \kappa u) \Big|_1^{-1} \\ &= \frac{q}{(2\pi)^2} \int_0^\infty d\kappa \frac{\sin(\kappa(t - t_0 + |\mathbf{x}|)) + \sin(\kappa(t_0 - t + |\mathbf{x}|))}{|\mathbf{x}| \kappa}. \end{aligned}$$

Note that I denote $|\mathbf{k}| = \kappa$ and $d\kappa$, integration over the radial coordinate, which should not be confused with the measure dk on the light cone. For $t = t_0$

$$\phi_c(0, \mathbf{x}) = \frac{q}{2\pi^2} \int_0^\infty d\kappa \frac{\sin \kappa |\mathbf{x}|}{\kappa |\mathbf{x}|} = \frac{q}{4\pi |\mathbf{x}|}.$$

For $\Delta t = t - t_0 \neq 0$ the field is symmetric in time, $\phi_c(\Delta t, \mathbf{x}) = \phi_c(-\Delta t, \mathbf{x})$. Assume $\Delta t > 0$,

$$\begin{aligned}\phi_c(\Delta t, \mathbf{x}) &= \frac{q}{(2\pi)^2} \int_0^\infty d\kappa \frac{\sin(\kappa(|\mathbf{x}| + \Delta t))}{|\mathbf{x}|\kappa} + \frac{q}{(2\pi)^2} \int_0^\infty d\kappa \frac{\sin(\kappa(|\mathbf{x}| - \Delta t))}{|\mathbf{x}|\kappa} \\ &= \frac{q}{(2\pi)^2} \frac{1}{|\mathbf{x}|} \frac{\pi}{2} + \frac{q}{(2\pi)^2} \frac{|\mathbf{x}| - \Delta t}{|\mathbf{x}|} \frac{|\mathbf{x}| - \Delta t}{|\mathbf{x}| - \Delta t} \int_0^\infty d\kappa \frac{\sin(\kappa(|\mathbf{x}| - \Delta t))}{\kappa(|\mathbf{x}| - \Delta t)} \\ &= \frac{q}{4\pi|\mathbf{x}|} \frac{1}{2} \left(1 + \text{sgn}(|\mathbf{x}| - \Delta t)\right) = \frac{q}{4\pi|\mathbf{x}|} \theta(|\mathbf{x}| - \Delta t).\end{aligned}\quad (568)$$

For $|\mathbf{x}| = \Delta t > 0$

$$\phi_c(\Delta t, \mathbf{x}) = \frac{q}{(2\pi)^2} \int_0^\infty d\kappa \frac{\sin 2\kappa|\mathbf{x}|}{\kappa|\mathbf{x}|} = \frac{q}{4\pi^2|\mathbf{x}|} \frac{\pi}{2} = \frac{q}{4\pi|\mathbf{x}|} \frac{1}{2} \quad (569)$$

For any t, t_0 ,

$$\phi_c(t - t_0, \mathbf{x}) = \frac{q}{4\pi|\mathbf{x}|} \theta(|\mathbf{x}| - |t - t_0|). \quad (570)$$

For fixed t and $|\mathbf{x}| \neq 0$

$$\lim_{t_0 \rightarrow \pm\infty} \phi_c(t - t_0, \mathbf{x}) = 0. \quad (571)$$

As we can see, the step function is defined by

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}. \quad (572)$$

The role of compensating field is similar to the one of compensating currents occurring in some approaches to quantum electrodynamics (cf. [52]). A difference is that compensating currents are introduced in an ad hoc manner, whereas our compensating field is a consequence of quantum dynamics.

B. Generalized Coulomb field

In order to understand why I call $\phi_{\text{gC}}(\mathbf{x})$ a generalized Coulomb field let us note that setting $I_0(\mathbf{k}) = 1$ one indeed arrives at the ordinary Coulomb field

$$-q \int \frac{d^3k}{(2\pi)^3 |\mathbf{k}|^2} \cos \mathbf{k} \cdot \mathbf{x} = -\frac{q}{4\pi|\mathbf{x}|} = \phi_{\text{C}}(\mathbf{x}) = -\phi_{\text{c}}(0, \mathbf{x}). \quad (573)$$

However, we know that $I_0(\mathbf{k}) = 1$ implies $|O_0(\mathbf{k})|^2 = 1$ (since $|O_0(\mathbf{k})|^2$ is the vacuum average of $I_0(\mathbf{k})$) which would lead to infrared divergences and all types of internal inconsistencies of the formalism. So, take

$$\begin{aligned} I_0(\mathbf{k}, N) &= \frac{1}{N} \left(I_0(\mathbf{k}, 1) \otimes I_0(1) \otimes \cdots \otimes I_0(1) + \cdots + I_0(1) \otimes \cdots \otimes I_0(1) \otimes I_0(\mathbf{k}, 1) \right), \\ \phi_{\text{gC}}(\mathbf{x}, N) &= \frac{1}{N} \left(\phi_{\text{gC}}(\mathbf{x}, 1) \otimes I_0(1) \otimes \cdots \otimes I_0(1) + \cdots + I_0(1) \otimes \cdots \otimes I_0(1) \otimes \phi_{\text{gC}}(\mathbf{x}, 1) \right), \\ \phi_{\text{gC}}(\mathbf{x}, 1) &= -q \int \frac{d^3 k}{(2\pi)^3 |\mathbf{k}|^2} \cos(\mathbf{k} \cdot \mathbf{x}) |\mathbf{k}\rangle \langle \mathbf{k}| \otimes 1 = -2q \int dk \frac{\cos \mathbf{k} \cdot \mathbf{x}}{|\mathbf{k}|} |\mathbf{k}\rangle \langle \mathbf{k}| \otimes 1 \\ &= -2q \frac{\cos \hat{\mathbf{k}} \cdot \mathbf{x}}{|\hat{\mathbf{k}}|} \otimes 1, \end{aligned} \quad (574)$$

$$\hat{\mathbf{k}} = \int dk \mathbf{k} |\mathbf{k}\rangle \langle \mathbf{k}|. \quad (575)$$

Of course, $\phi_{\text{gC}}(\mathbf{x}, N)$ is a field *operator* commuting with all elements of HOLA (the same modification of Coulomb's law was found in full electrodynamics in [27]). Vacuum averages of $\phi_{\text{gC}}(\mathbf{x}, N)$ are ordinary functions that can be compared with $\phi_{\text{C}}(\mathbf{x}, N)$.

Let us thus consider the average evaluated in a vacuum state $|O_0, N\rangle$,

$$\langle O_0, N | \phi_{\text{gC}}(\mathbf{x}, N) | O_0, N \rangle = -q \int \frac{d^3 k}{(2\pi)^3 |\mathbf{k}|^2} |O_0(\mathbf{k})|^2 \cos \mathbf{k} \cdot \mathbf{x}. \quad (576)$$

Probability $|O_0(\mathbf{k})|^2$ occurs in place where in the usual approach one might, if needed, put a cutoff function $\chi(\mathbf{k})$. Cutoff functions are assumed to vanish for large and small \mathbf{k} but somewhere in between infrared and ultraviolet regimes achieve their maximal value 1. It is obvious, however, that the maximal value of $|O_0(\mathbf{k})|^2$ is greater from 0, but may be different from unity. It follows that $|O_0(\mathbf{k})|^2$ itself cannot be regarded as the usual cutoff function. The latter is obtained if one divides $|O_0(\mathbf{k})|^2$ by its maximal value Z , say.

We know that Poincaré transformations transform $|O_0(\mathbf{k})|$ into $|O_0(\mathbf{L}^{-1}\mathbf{k})|$. Since the maximum

$$Z = \max_{\mathbf{k}} \{|O_0(\mathbf{k})|^2\} = \max_{\mathbf{k}} \{|O_0(\mathbf{L}^{-1}\mathbf{k})|^2\} \quad (577)$$

is Poincaré invariant we can define in Poincaré covariant way the normalized “cutoff function”

$$\chi_0(\mathbf{k}) = |O_0(\mathbf{k})|^2 / Z, \quad 0 \leq \chi_0(\mathbf{k}) \leq 1. \quad (578)$$

Accordingly,

$$\langle O_0, N | \phi_{\text{gC}}(\mathbf{x}, N) | O_0, N \rangle = -qZ \int \frac{d^3 k}{(2\pi)^3 |\mathbf{k}|^2} \chi(\mathbf{k}) \cos \mathbf{k} \cdot \mathbf{x}, \quad (579)$$

suggesting that qZ is the physical charge. This statement is *almost* true but all the examples studied in the literature so far (cf. [20, 27]) show that the correct physical parameter that should be compared with experiment is $q^2Z = (q\sqrt{Z})^2 = q_{\text{ph}}^2$. At this stage I propose the readers just to take this statement for granted, without proof. In other words, if we want to test the average with respect to the classical Coulomb field one should compare $-q_{\text{ph}}^2/(4\pi|\mathbf{x}|)$ with

$$q\langle O_0, N | \phi_{\text{gC}}(\mathbf{x}, N) | O_0, N \rangle = -q_{\text{ph}}^2 \int \frac{d^3k}{(2\pi)^3 |\mathbf{k}|^2} \chi(\mathbf{k}) \cos \mathbf{k} \cdot \mathbf{x}, \quad (580)$$

where $q_{\text{ph}} = q\sqrt{Z}$ is the *renormalized* (i.e. physical) charge and Z is the renormalization constant (analogous to Z_3 known from textbooks [52]). Let us also note here that condition (565) makes

$$\langle O_0, N | \phi_{\text{gC}}(\mathbf{0}, N) | O_0, N \rangle = -q \int \frac{d^3k}{(2\pi)^3 |\mathbf{k}|^2} |O_0(\mathbf{k})|^2 \quad (581)$$

finite.

To have a feel of general properties of $q\langle O_0, N | \phi_{\text{gC}}(\mathbf{x}, N) | O_0, N \rangle$ let us take a rotationally invariant $|O_0(\mathbf{k})|^2$ which vanishes for $0 \leq |\mathbf{k}| \leq k_1$ and $|\mathbf{k}| \geq k_2$ but is otherwise constant (equal to some Z). Then Z can be then computed from

$$1 = \int dk |O_0(\mathbf{k})|^2 = 4\pi Z \int_{k_1}^{k_2} \frac{d\kappa \kappa}{2(2\pi)^3} = \frac{Z(k_2^2 - k_1^2)}{8\pi^2}, \quad (582)$$

$$Z = \frac{8\pi^2}{k_2^2 - k_1^2}, \quad (583)$$

$$q_{\text{ph}} = q\sqrt{Z} = q \frac{2\sqrt{2}\pi}{\sqrt{k_2^2 - k_1^2}}. \quad (584)$$

Now

$$\begin{aligned} q\langle O_0, N | \phi_{\text{gC}}(\mathbf{x}, N) | O_0, N \rangle &= -\frac{2\pi q_{\text{ph}}^2}{(2\pi)^3} \int_{k_1}^{k_2} d\kappa \int_0^\pi d\theta \sin \theta \cos(\kappa \cos \theta |\mathbf{x}|) \\ &= \frac{q_{\text{ph}}^2}{4\pi^2} \int_{k_1}^{k_2} d\kappa \int_1^{-1} du \cos(\kappa |\mathbf{x}| u) \\ &= -\frac{q_{\text{ph}}^2}{2\pi^2} \int_{k_1}^{k_2} d\kappa \frac{\sin \kappa |\mathbf{x}|}{\kappa |\mathbf{x}|} \\ &= -\frac{q_{\text{ph}}^2}{4\pi |\mathbf{x}|} \frac{\text{Si}(k_2 |\mathbf{x}|) - \text{Si}(k_1 |\mathbf{x}|)}{\pi/2}. \end{aligned} \quad (585)$$

The sine integral function (see Fig. 7)

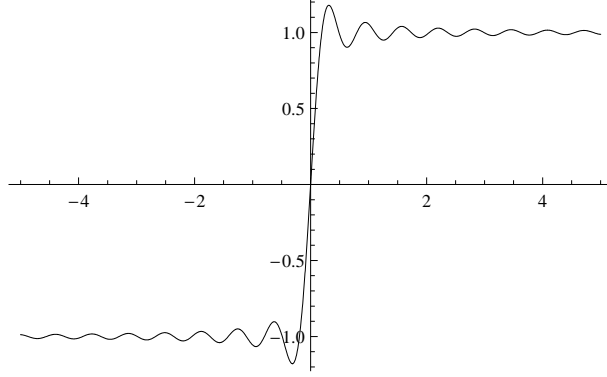


FIG. 7: Normalized sine integral $\frac{2}{\pi}\text{Si}(kx)$ for small x behaves as $2kx/\pi$ but for large $|x|$ approaches 1 independently of k (on the plot $k = 10$). This is why $\langle O_0, N | \phi_{\text{gC}}(\mathbf{x}, N) | O_0, N \rangle$ is finite at the origin and for large $|\mathbf{x}|$ decays faster than $1/|\mathbf{x}|$. If k_1 were exactly 0 (no infrared cutoff) then $\langle O_0, N | \phi_{\text{gC}}(\mathbf{x}, N) | O_0, N \rangle$ would asymptotically approach the Coulomb field. However, $k_1 = 0$ would be in contradiction with $\lim_{\mathbf{k} \rightarrow \mathbf{0}} O_0(\mathbf{k}) = 0$ which we assume.

$$\text{Si}(x) = \int_0^x d\kappa \frac{\sin \kappa}{\kappa} \quad (586)$$

has series expansion

$$\text{Si}(x) = x - x^3/18 + x^5/600 - x^7/35280 + x^9/3265920 + \dots \quad (587)$$

so

$$\lim_{x \rightarrow 0} \frac{\text{Si}(k_2 x) - \text{Si}(k_1 x)}{x} = k_2 - k_1. \quad (588)$$

Another important property of $\text{Si}(x)$ is the smallest solution of

$$\text{Si}(\infty) - \text{Si}(x) = 0 \quad (589)$$

which is numerically found to be $x \approx 1.92645$. What this physically means is that for a very large k_2 (a property one expects on physical grounds) the potential

$$-\frac{q_{\text{ph}}^2}{4\pi|\mathbf{x}|} \frac{\text{Si}(k_2|\mathbf{x}|) - \text{Si}(k_1|\mathbf{x}|)}{\pi/2} \quad (590)$$

changes sign if $|\mathbf{x}| \approx 1.92645/k_1$. Oscillating changes of sign are implied by oscillations of $\text{Si}(x)$ depicted at Fig. 7. It is interesting that suggestions of analogous changes of sign have been seriously considered in cosmology and astrophysics in the context of gravitational Newton law (see [54] for a systematized guide over the literature of the subject).

Current experimental data show that exact Coulomb law is indistinguishable, up to $|\mathbf{x}|$ as large as the Earth-Sun distance, from the Yukawa law

$$-\frac{q_{\text{ph}}^2}{4\pi|\mathbf{x}|}e^{-|\mathbf{x}|/\lambda} \quad (591)$$

where λ is of the order of the Earth-Moon distance (more than 300000 km, corresponding to the Compton wavelength of a particle whose mass is $10^{-48}\text{g} < m < 10^{-47}\text{g}$, cf. the review [55]). The experiments are supposed to test the value of photon's rest mass — the tacit assumption being that for massless photons and pointlike sources one necessarily gets the Coulomb law. However, we have just seen that the result depends also on quantization procedures — in my formalism the field is massless and the source is pointlike, and yet the Coulomb law is derived in a generalized form.

Anyway, returning to the data, the Yukawa potential (591) effectively deviates from the Coulomb law for large $|\mathbf{x}|$, a behavior controlled in my formula by the infrared cutoff k_1 . The data from [55] show that varying λ between, roughly, $\lambda_{\min} = 300000$ km and infinity we remain within the experimental uncertainty even for $|\mathbf{x}|$ as large as some 10^9 km. The error of determining the exact Coulomb law thus can be defined as

$$\Delta(\mathbf{x}) = \frac{1}{|\mathbf{x}|} - \frac{e^{-|\mathbf{x}|/\lambda_{\min}}}{|\mathbf{x}|}. \quad (592)$$

The scale of infrared cutoff is determined by k_1 , which can be estimated on the basis of

$$\frac{e^{-|\mathbf{x}|/\lambda_{\min}}}{|\mathbf{x}|} - \Delta(\mathbf{x}) = 2\frac{e^{-|\mathbf{x}|/\lambda_{\min}}}{|\mathbf{x}|} - \frac{1}{|\mathbf{x}|} \leq \frac{1}{|\mathbf{x}|} \frac{\text{Si}(\infty) - \text{Si}(k_1|\mathbf{x}|)}{\pi/2} = \frac{1}{|\mathbf{x}|} - \frac{2}{\pi|\mathbf{x}|}\text{Si}(k_1|\mathbf{x}|) \leq \frac{1}{|\mathbf{x}|}$$

at least for $0 \ll |\mathbf{x}| < 10^9$ km. So,

$$2\frac{e^{-|\mathbf{x}|/\lambda_{\min}}}{|\mathbf{x}|} - 2\frac{1}{|\mathbf{x}|} \leq -\frac{2}{\pi|\mathbf{x}|}\text{Si}(k_1|\mathbf{x}|) \leq 0,$$

and finally

$$0 \leq \pi(1 - e^{-|\mathbf{x}|/\lambda_{\min}}) - \text{Si}(k_1|\mathbf{x}|). \quad (593)$$

Fig. 9 shows that for $k_1 = 2/\lambda_{\min}$ (or smaller) inequality (593) is satisfied for *all* values of $|\mathbf{x}|$.

The conclusion is that even the most precise tests of the Coulomb law available so far do not contradict the possibility that $I_0(\mathbf{k}) \neq 1$. $O_0(\mathbf{k})$ may be a nontrivial function that tends to 0 with $\mathbf{k} \rightarrow \mathbf{0}$. The analysis I have presented is based on a concrete form of $O_0(\mathbf{k})$

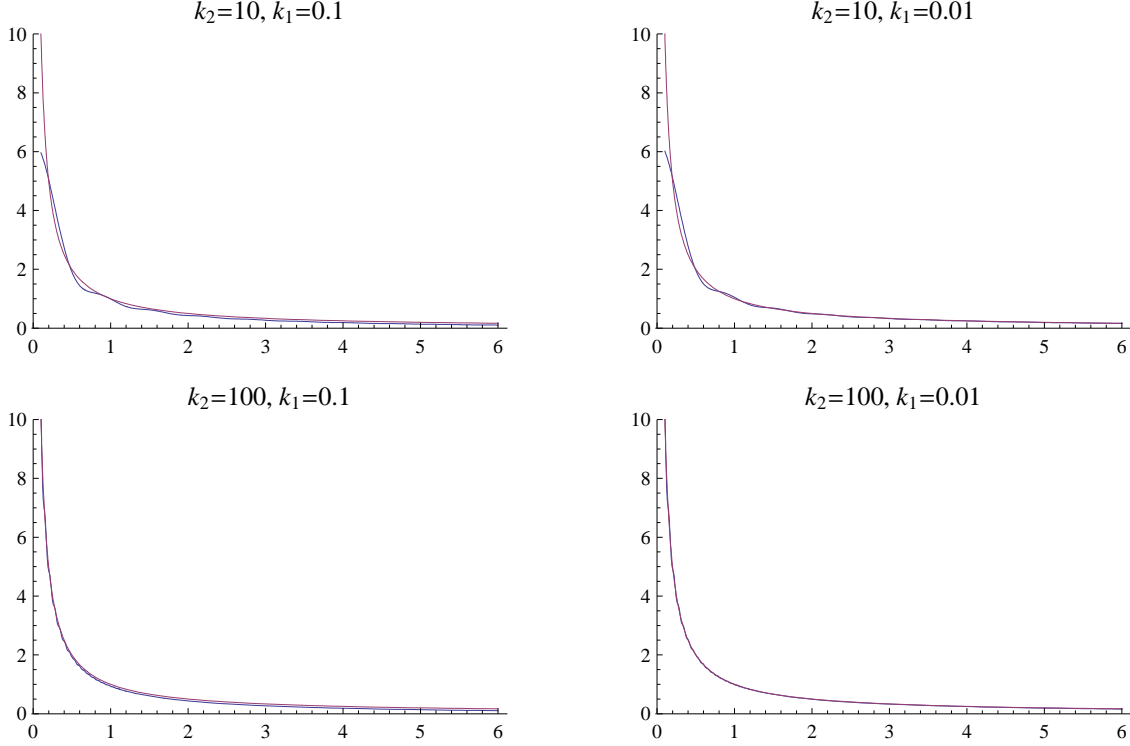


FIG. 8: Comparison of $1/x$ with $2[\text{Si}(k_2x) - \text{Si}(k_1x)]/(\pi x)$ for arbitrarily chosen k_1 and k_2 .

but one should not expect drastically different predictions if $O_0(\mathbf{k})$ decayed in the infrared regime in a smoother way.

Another important test of the Coulomb law would be the energy spectrum of hydrogen-like atoms with the *operator* potential $\phi_{\text{gC}}(\mathbf{x})$. I hope to return to this problem later on in these notes.

C. “Photon statistics”

Averages of generalized Coulomb field did not depend on the parameter N of the representation of HOLA. N becomes essential for the n -“scalar photon” statistics.

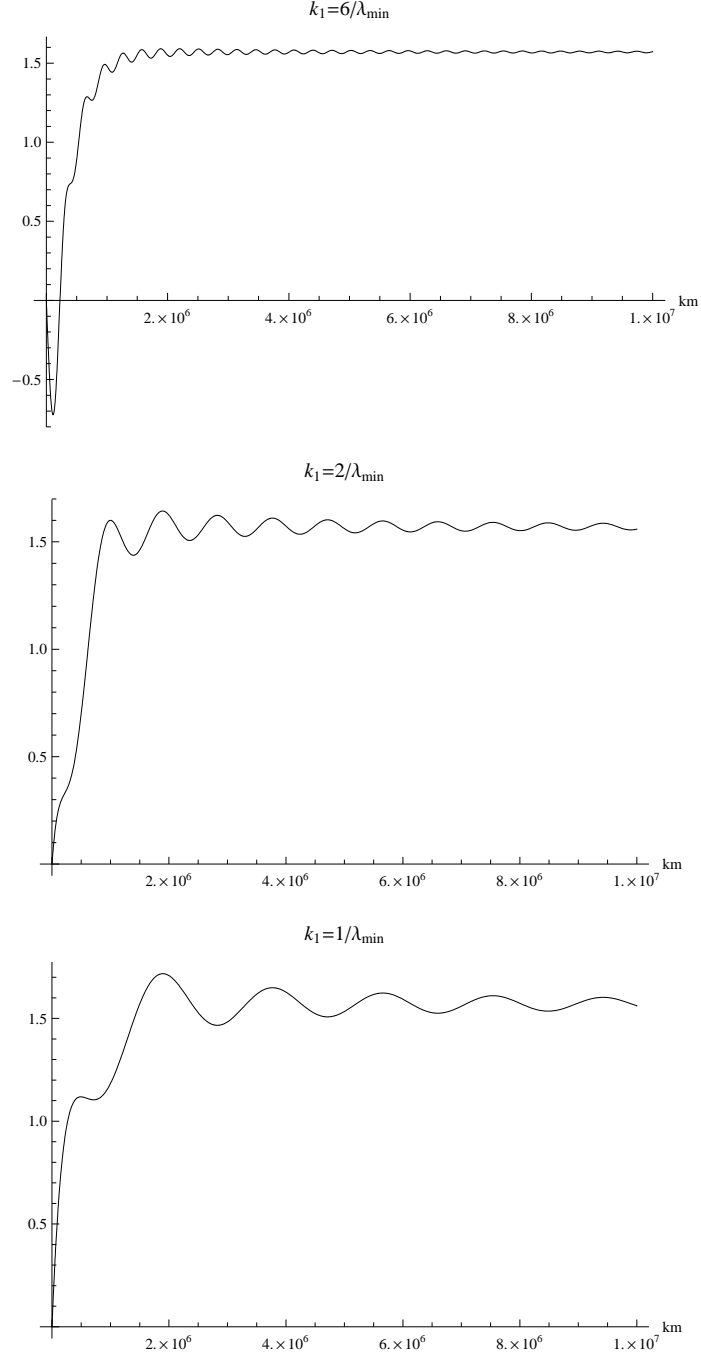


FIG. 9: Plots of $\pi(1 - e^{-|\mathbf{x}|/\lambda_{\min}}) - \text{Si}(k_1|\mathbf{x}|)$ for $\lambda_{\min} = 300000$ km and different values of k_1 . For $k_1 = 2/\lambda_{\min}$, or smaller, inequality (593) is satisfied for all $|\mathbf{x}|$.

Let us begin with the unitary operator

$$\begin{aligned}
e^{-i \int_{t_0}^t d\tau V(\tau)} &= \exp \left(-iq \int_{t_0}^t d\tau \phi(\tau, \mathbf{0}, N) \right) \\
&= \exp \left(-iq \int_{t_0}^t d\tau \int dk \left(c(\mathbf{k}, N) e^{-i|\mathbf{k}|\tau} + c(\mathbf{k}, N)^\dagger e^{i|\mathbf{k}|\tau} \right) \right) \\
&= \exp \left(-iq \int dk \left(c(\mathbf{k}, N) \frac{e^{-i|\mathbf{k}|t} - e^{-i|\mathbf{k}|t_0}}{-i|\mathbf{k}|} + c(\mathbf{k}, N)^\dagger \frac{e^{i|\mathbf{k}|t} - e^{i|\mathbf{k}|t_0}}{i|\mathbf{k}|} \right) \right) \\
&= \exp \int dk \left(\alpha_{t,t_0}(\mathbf{k}) c(\mathbf{k}, N)^\dagger - \overline{\alpha_{t,t_0}(\mathbf{k})} c(\mathbf{k}, N) \right) \\
&= \mathcal{D}_0(\alpha_{t,t_0}, N).
\end{aligned} \tag{594}$$

This is a coherent-state displacement operator with

$$\alpha_{t,t_0}(\mathbf{k}) = q \frac{e^{i|\mathbf{k}|t_0} - e^{i|\mathbf{k}|t}}{|\mathbf{k}|}. \tag{595}$$

Repeating calculations analogous to those that led to (184) we obtain statistics of “scalar photons” produced by a pointlike charge, with vacuum initial condition at $t = t_0$,

$$\begin{aligned}
p(n, N) &= \langle O_0, N | \exp \left(- \int dk |\alpha_{t,t_0}(\mathbf{k})|^2 I_0(\mathbf{k}, N) \right) \frac{1}{n!} \left(\int dk_1 |\alpha_{t,t_0}(\mathbf{k}_1)|^2 I_0(\mathbf{k}_1, N) \right)^n | O_0, N \rangle \\
&= \frac{1}{n!} \frac{d^n}{d\lambda^n} \langle O_0, N | \exp \left(\lambda \int dk |\alpha_{t,t_0}(\mathbf{k})|^2 I_0(\mathbf{k}, N) \right) | O_0, N \rangle \Big|_{\lambda=-1}.
\end{aligned} \tag{596}$$

Generating function occurring in (596) can be further transformed, leading to Kolmogorov-Nagumo average

$$\langle O_0, N | \exp \left(\lambda \int dk |\alpha_{t,t_0}(\mathbf{k})|^2 I_0(\mathbf{k}, N) \right) | O_0, N \rangle = \left(\int dk |O_0(\mathbf{k})|^2 e^{\lambda \frac{1}{N} |\alpha_{t,t_0}(\mathbf{k})|^2} \right)^N. \tag{597}$$

The limiting case

$$\lim_{N \rightarrow \infty} \left(\int dk |O_0(\mathbf{k})|^2 e^{\lambda \frac{1}{N} |\alpha_{t,t_0}(\mathbf{k})|^2} \right)^N = \exp \left(\lambda \int dk |O_0(\mathbf{k})|^2 |\alpha_{t,t_0}(\mathbf{k})|^2 \right) \tag{598}$$

is a regularized Poisson distribution. Explicitly,

$$\begin{aligned}
|\alpha_{t,t_0}(\mathbf{k})|^2 &= q^2 \frac{2 - e^{i|\mathbf{k}|(t_0-t)} - e^{-i|\mathbf{k}|(t_0-t)}}{|\mathbf{k}|^2} = q^2 \frac{2 - 2 \cos(|\mathbf{k}|(t - t_0))}{|\mathbf{k}|^2} \\
&= q^2 \frac{\sin^2(|\mathbf{k}|(t - t_0)/2)}{(|\mathbf{k}|/2)^2}
\end{aligned} \tag{599}$$

and

$$\begin{aligned}
\exp \left(\lambda \int dk |O_0(\mathbf{k})|^2 |\alpha_{t,t_0}(\mathbf{k})|^2 \right) &= \exp \left(\lambda q^2 \int dk |O_0(\mathbf{k})|^2 \frac{\sin^2(|\mathbf{k}|(t - t_0)/2)}{(|\mathbf{k}|/2)^2} \right) \\
&= \exp \left(\lambda q_{\text{ph}}^2 \int dk \chi(\mathbf{k}) \frac{\sin^2(|\mathbf{k}|(t - t_0)/2)}{(|\mathbf{k}|/2)^2} \right)
\end{aligned} \tag{600}$$

Let us take a closer look at the exponent in (600). Since,

$$0 \leq \frac{\sin^2(|\mathbf{k}|\Delta t/2)}{(|\mathbf{k}|/2)^2} = \Delta t^2 \frac{\sin^2(|\mathbf{k}|\Delta t/2)}{(|\mathbf{k}|\Delta t/2)^2} \leq \Delta t^2 \quad (601)$$

the integral is finite at least for finite $\Delta t = t - t_0$,

$$\int dk |O_0(\mathbf{k})|^2 \frac{\sin^2(|\mathbf{k}|(t - t_0)/2)}{(|\mathbf{k}|/2)^2} \leq (t - t_0)^2 \int dk |O_0(\mathbf{k})|^2 = (t - t_0)^2. \quad (602)$$

In the ordinary approach we would obtain the same formula (600) but with $|O_0(\mathbf{k})|^2 = 1$, i.e.

$$\int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} \frac{\sin^2(|\mathbf{k}|\Delta t/2)}{(|\mathbf{k}|/2)^2} = \frac{16\pi}{2(2\pi)^3} \int_0^\infty d\kappa \frac{\sin^2(\kappa\Delta t/2)}{\kappa}. \quad (603)$$

The latter integral vanishes for $\Delta t = 0$, but becomes divergent for $\Delta t > 0$. More precisely, for $0 < k_1 < k_2$,

$$\int_{k_1}^{k_2} d\kappa \frac{\sin^2 \kappa}{\kappa} = \frac{\text{Ci}(2k_1) - \ln k_1 - \text{Ci}(2k_2) + \ln k_2}{2}. \quad (604)$$

The infrared limit is finite,

$$\lim_{k_1 \rightarrow 0} (\text{Ci}(2k_1) - \ln k_1) = \gamma + \ln 2 \quad (605)$$

(Ci and $\gamma \approx 0.577216$ are, respectively, the cosine integral function and the Euler constant). The infinity comes from the ultraviolet divergency of $\ln k_2$. An infrared divergency would, however, also appear if the charge was accelerating (for details of standard calculations cf. [52] and Sec. 4-1-2 in [56]; analogous results for N -representations of HOLA and true electromagnetic fields are described in detail in [27]). One concludes that the standard approach to pointlike classical sources leads to mathematically ill defined objects such as $e^{\lambda\infty}$ and thus is mathematically inconsistent.

In “my” formalism one finds that probability of finding n “scalar photons” is given by the vacuum average of POVM

$$\begin{aligned} \Pi(n, N) &= \exp\left(-\int dk |\alpha_{t,t_0}(\mathbf{k})|^2 I_0(\mathbf{k}, N)\right) \frac{1}{n!} \left(\int dk_1 |\alpha_{t,t_0}(\mathbf{k}_1)|^2 I_0(\mathbf{k}_1, N)\right)^n \\ &= \frac{1}{n!} \frac{d^n}{d\lambda^n} \exp\left(\lambda \int dk |\alpha_{t,t_0}(\mathbf{k})|^2 I_0(\mathbf{k}, N)\right) \Big|_{\lambda=-1}, \end{aligned} \quad (606)$$

$$\sum_{n=0}^{\infty} \Pi(n, N) = I_0(N). \quad (607)$$

This POVM is uniquely and well defined, but its average depends on the choice of vacuum, of course.

Now let us discuss the issue of the limit $t_0 \rightarrow \pm\infty$. Let us return to

$$\begin{aligned}
\int dk |O_0(\mathbf{k})|^2 |\alpha_{t,t_0}(\mathbf{k})|^2 &= q^2 \int dk |O_0(\mathbf{k})|^2 \frac{2 - 2 \cos(|\mathbf{k}|(t - t_0))}{|\mathbf{k}|^2} \\
&= 2q^2 \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} \frac{|O_0(\mathbf{k})|^2}{|\mathbf{k}|^2} \\
&\quad - 2q^2 \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} |O_0(\mathbf{k})|^2 \frac{\cos(|\mathbf{k}|t) \cos(|\mathbf{k}|t_0)}{|\mathbf{k}|^2} \\
&\quad - 2q^2 \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} |O_0(\mathbf{k})|^2 \frac{\sin(|\mathbf{k}|t) \sin(|\mathbf{k}|t_0)}{|\mathbf{k}|^2}. \quad (608)
\end{aligned}$$

If the first integral

$$\int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} \frac{|O_0(\mathbf{k})|^2}{|\mathbf{k}|^2}$$

is finite (compare (442)) then $|O_0(\mathbf{k})|^2/|\mathbf{k}|^3$ satisfies assumptions of the Riemann-Lebesgue lemma, and

$$\lim_{t_0 \rightarrow \pm\infty} \int dk |O_0(\mathbf{k})|^2 |\alpha_{t,t_0}(\mathbf{k})|^2 = 2q^2 \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} \frac{|O_0(\mathbf{k})|^2}{|\mathbf{k}|^2}.$$

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